Mild Solutions for Semi-Linear Fractional Evolution Systems with Nonlinear Integral Boundary Conditions: Existence and Uniqueness

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Abstract
Fractional evolution equations, focusing on infinite dimensions, are methods for describing processes with memory and processes with hereditary types. This paper constructs a solvability framework that differs from previous neutral fractional evolution equations, where operator-valued multi-point non-local boundary problems were dependent on non-linear integral non-local boundary problems in abstract Banach spaces. We include the Caputo fractional derivative with order α more than 0 but less than 1. The linear part is assumed to generate a strongly continuous semigroup. The author proves that mild solutions(MS)exist by transforming the problem into an equal integral equation and applying the Krasnoselskii's Fixed Point Theorem. This also gives a different way of solving the previous paper's problem with Sadovskii's approach. Under some conditions that are similar to the Lipschitz condition, the author proves that, by the Banach contraction principle, the Mild solution becomes exclusive. Therefore, the present work provides a different mathematical approach for the neutral nonlinear fractional evolution system, considering the boundary structure, the system architect, the fixed-point technique different from the first one, and Sadovskii's theorem. For the theoretical results to be True, a sample numeric case is solved in MATLAB to demonstrate that the solutions behave according to the fractional order parameters of the model, and the results show that the approximate solutions converge.The results received strengthen the nonlocal fractional evolution equations and nonlocal interacting fractional evolution equations with respect to their boundary interactions and allow us to identify concrete directions for further analytical and numerical investigations.

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I. INTRODUCTION

Fractal Evolution Equations are developed in many other disciplines, such as science, engineering, control theory, and modeling memory or hereditary systems, as opposed to classical integer-order modeling. Semilinear fractal evolution systems, in the setting of infinite-dimensional dynamical systems theory, offer a starting point for the analysis of fractal spaces.Thus far, in a sense, applying fractal evolution equations to neutral fractional systems and operator-valued multipoint boundary/initial conditions is classical.

In most practical cases, however, boundary interactions are a lot more complicated owing to the time-related history of the state variable, as opposed to having a finite state at the boundary. In this sense, we have noted a more systematic reasoning for the analysis of nonlocal boundary conditions in order to address the issue at hand. Generally, these boundary conditions are more appropriate, but they necessitate a greater deal of effort to ascertain the presence of uniquesolutions.

This article introduces new ideas about Caputo fractional derivatives with nonlinear (NL) integral boundary conditions in a nonscalar abstract Banach space, as well as SL fractional evolution equations. It will be assumed that the linear part of the problem defines a strongly continuous semigroup, thus providing the necessary conditions for well-posedness of the concerned evolution. After reformulating the boundary value problem (BVP) in an equal form as an essential equation, the fixed point approach will be used in order to obtain some solvability results. Specifically, it is the Banach contraction principle that addresses the individuality in a Lipschitz-type condition, while the presence of mild solutions (MS's) will be dealt with within the framework of the Krasnoselskii's fixed point theorem (FPT).

The main targets of the research are as follows:

- To develop the abstract Banach space framework for a fractional evolution semilinear system with nonlinear integral-type boundary conditions.
- To recast the problem in question into an equivalent integral (mild) equation, relying on semigroup theory.
- To employ the Krasnoselskii FPT in order to establish the presence of MS's.
- To provide conditions under which, by the Banach contraction principle, the uniqueness of solutions is ensured.
- To analyze the interboundary nonlinear configurations.
- Lastly, to integrate some of the theoretical findings with a numerical convergence analysis.

The structure of the rest of the paper is that the 2nddivision gives a review of the literature. 3rd division describes the necessary background concepts and the underlying mathematical theories relevant to the analysis, namely, semigroup theory and fractional calculus. This is followed by problem formulation, wherein the problem is expressed in an equivalent formulation of an integral. The presence and exclusivity of solutions are established by applying appropriate FTP. The theoretical results are then demonstrated with a numerical example. To sum up, the research unites the key findings and reveals possible directions of the investigation.

II. RELATED WORKS

There has recently been considerable interest in optimal control theory in infinite dimensions and controllability, as well as the study of evolution fractional semilinear equations (SL eqns). The distributed parameter systems have become more analytically challenging with the inclusion of fractional derivatives, delays, stochastic perturbations, and non-local conditions. Strengthening the controllability, stability, and optimal control problems within the generalized fractional framework, as well as the existence theory, has been the focus of several recent authors in this area.

Zerbib et al. (2026) generalized the Laplace transforms and discovered the analytical solutions of semilinear (SL) fractional evolution systems with variable-order Phi-Caputo derivatives. Feng and Chen (2024) provided the first existence results for non-autonomous nonlocal progress equations with superlinear growth nonlinearities under relaxed growth conditions. Li et al. (2024) utilized the non-compactness principle to establish the existence of solutions for fractional delayed evolution equations. Wu et al. (2024) gave the results of the existence and the periodicity of neutral measure functional differential equations(DE) with infinite delays. Gou and Shi (2024) used the monotone iterative method to time-fractionally discretize the DE of many terms and generalized the classical fixed-point method to systems where the measure is a fraction.

There has been progress in researching the controllability of fractional and measure evolution systems. Via the resolvent operator theory, Chen et al. (2020) approached the problem of approximation. Controllability up to an arbitrary precision for fractional evolution equations under nonlocal constraints. Mahmudov (2020) integrated stochastic effects in the control theory of semilinear fractional stochastic integro-DE. Liu and Liu (2023) considered the backward trace of controllability for fractional quantity evolution systems with state-dependent delays and nonlocal conditions under compactness assumptions. Tajani et al. (2024) investigated the boundary controllability of Riemann-Liouville fractional semilinear equations, and Nunes (2021) studied exact periphery controllability and the energy decay rate for coupled wave systems. Xue et al. (2024) introduced the notion of linear quadratic optimal control for distributed hyperbolic systems and developed control methods for 1st order hyperbolic partial differential equations.

The optimality of SL equations. and their fractional variants have been studied extensively. In this regard, Casas et al. (2017) focused on the optimum control of SL parabolic equations with bounded variation control and established some continuity and regularity results. Krug et al. (2021) described, improved and made more computational SL hyperbolic equations optimal control problems within the framework of time-domain decomposition. Otárola (2022) provided optimality conditions, convergence results and error estimations for fractional SL optimal control problems. In the same manner, Casas and Wachsmuth (2023) examined the control problem of the existence of SL PDEs under more general assumptions. Cheddour and Elazzouzi (2023) considered optimal feedback control of infinite-dimensional SL systems with distributed delays, and Johnson and Vijayakumar (2023) investigated the idealcontroller of fractional stochastic delay integro-DEs.

The theory of optimal fractional control has also been applied to the field of epidemiology and networks. Rashid et al. (2025) formulated the first hybrid fractional model of monkeypox disease with exogenous factors and solved it via optimal control. Hou et al. (2025) studied the optimal control of activity-based stochastic diffusion networks with rumor spreading, incorporating the memory effect in the diffusion process of the network. Diop et al. (2024) extended the fractional stochastic approach and applied it to the study of stochastic development equations in Banach spaces, weighted, pseudo-asymptotically periodic, and driven by fractional Brownian motion.

In spite of at least some knowledge being available regarding theory, fractional semilinear systems, controllability, optimal control, and several of the issues, including the fractals, infinite delays, measures, and the interplay between them, the existence of a unifying theory is needed. In addition, the strong feedback of the evolution systems of fractional measures and the almost sure weak compactness associated with limit theorems of convergence of the systems remain unexamined. The above issues should create the necessity of a more comprehensive theoretical framework that combines the controllability, stability, and optimum control of fractional SL evolution systems with delays and measures.

III. PRELIMINARIES

Let $C([0,T],X)$ be the Banach space of all continuous functions from the interval [0, T] into a real Banach space X endowed with a norm $\|\cdot\|$denoted by.

$\|x\|_\infty = \sup_{t\in[0,T]} \|x(t)\|$ (1)

Assume $0 < \alpha < 1$. The Caputo fractional derivative of order (α) for a function ($x: [0,T] \to X$) is denoted by

${}^{C}D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s)}{(t-s)^{\alpha}} ds$ (2)

It is assumed that the right-hand side $\Gamma(\cdot)$ is the Gamma function, which exists.

In $A: D(A) \subset X \to X$, a linear operator, the operator (A) generates a sturdilyincessant semigroup $\{S(t)\}_{t\geq 0}$on (X) if the following conditions hold:

$$S(0) = I, S(t+s) = S(t)S(s), \text{ and } \lim_{t\to 0^+} S(t)x = x \ \forall x \in X (3)$$

When combined with the fractional system, the solution operator is assumed to be of the following form of a fractional evolution family.

$T(t) = \int_0^\infty \phi_\alpha(\theta) S(t^\alpha \theta) d\theta$ (4)

where $\phi_\alpha(\theta)$is a probability density function associated with the Mittag-Leffler function. An MS of the fractional evolution system (FES) is a continuous function $x \in C([0,T], X)$that satisfies the integral equation:

$x(t) = S(t)x_0 + \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, x(s)) ds$ (5)

From equation (5) for all $t \in [0,T]$, where $x_0 \in X$ and $f: [0,T] \times X \to X$ is a given nonlinear function.

To state the existence of solutions, Krasnoselskii'sFPT is used.

Krasnoselskii'sFPT: Let (B) be an open, compact and convex subset of a Banach space (X). Let (F) and (G) be the operators that map (B) to (X) such that:

- F is a contraction,
- G is continuous and compact,
- $F(x) + G(x) \in B$ for all $x \in B$.

Then, we see x ∈ B, such that

$x = F(x) + G(x)$ (6)

To show that this is unique, we apply the Banach contraction principle. Banach Contraction Principle: Let H: X→X be an operator that satisfies

$\|H(x) - H(y)\| \leq k\|x - y\|, 0 < k < 1,$ (7)

For all x,y ∈X. Then, H has a unique fixed point in X. The following sections will show how these concepts will be used to show the existence and uniqueness of MS's of the given semilinear FES with nonlinear integral boundary conditions.

IV. PROBLEM FORMULATION

Let (X) be a real Banach space and consider the system of semilinear fractional evolution with Caputo fractional derivative of order ($0 < \alpha < 1$)

$$^{C}D_t^{\alpha}x(t) = Ax(t) + f\big(t, x(t)\big), t \in [0, T] \quad (8)$$

where $A: D(A) \subset X \to X$ is a linear operator and $f: [0, T] \times X \to X$ is a NE function.
The system is augmented with an NE integral boundary condition given by

$$x(0) = \int_0^T g\big(s, x(s)\big)ds \quad (9)$$

$g: [0, T] \times X \to X$ is a NE boundary operator.
In order for the operator A to define a sturdilyincessant semigroup $S(t)$on ($t \geq 0$), which makes a linear evolution problem well-posed, the operator A needs to act in a certain way. The NL,$f(t, x(t))$ has, in a sense, the appropriate continuity and growth conditions for the problem to be well-posed in the sense of Banach space. In this case, we are trying to formulate a fractional dynamical system with the memory effects modelled via the Caputo derivative. It has a nonlocal boundary condition, which is to say that the solution value $x(0)$ is determined by the entire trajectory of the solution $x(t)$ over the interval $[0, T]$. The complexity is that, when compared with classical initial value problems, the nonlinear part of the boundary condition affects the overall structure, making the analysis more challenging. However, one can use semigroup theory to bring the system to an equivalent integral formulation and streamline the analysis. A MS to the system is a function $x \in C([0, T], X)$ which fulfils,

$$x(t) = S(t)x(0) + \int_0^t (t-s)^{\alpha-1}T(t-s)f(s, x(s))ds, \quad (10)$$

$$for \quad t \in [0, T]$$

where S(t) is a sturdilyunceasing semigroup of A, and T(t) is the associated fractional operator. If the NL integral boundary condition is exchanged for the BVP in (10), then

$$x(t) = S(t)\left(\int_0^T g\big(s, x(s)\big)ds\right) +$$

$$\int_0^t (t-s)^{\alpha-1}T(t-s)f\big(s, x(s)\big)ds \quad (11)$$

Thus, the fractional differential system, along with the nonlinear integral boundary condition, can be reduced to an operator equation in the Banachspace $C([0, T], X)$.
Expressing the operator $\mathcal{F}: C([0, T], X) \to C([0, T], X)$ by

$$(\mathcal{F}x)(t) = S(t)\left(\int_0^T g\big(s, x(s)\big)ds\right) + \int_0^t (t-s)^{\alpha-1}T$$

$$(t-s)f\big(s, x(s)\big)ds \quad (12)$$

Then the problem reduces to finding the function $x \in C([0, T], X)$such that

$$x(t) = (\mathcal{F}x)(t), t \in [0, T] \quad (13)$$

To use any of the FTP's, the operator F is divided into two parts.

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 (14)$$

In this decomposition, one of the operators is assumed to be a contraction, and the other operator is assumed to be continuous and compact on a closed and bounded convex subset of C[0, T], X. By this decomposition, the fixed point theorem of Krasnoselskii can be used to demonstrate the presence of MS's. Additionally, with suitable Lipschitz-type conditions on the NL functions f and g, the operator F can be shown to satisfy a contraction condition.

$$\|\mathcal{F}x - \mathcal{F}y\| \leq k\|x - y\|, 0 < k < 1, \quad (15)$$

Equation (15) states the exclusivity of the MS according to the Banach contraction principle.

V. MAIN RESULTS

The assumptions (H1)-(H9) are crucial in demonstrating the presence and exclusivity of MS's to the semilinear FES with nonlinear integral boundary conditions. Each of the hypotheses makes F a well-defined operator that fulfills the requirements to use the fixed point theorems in the Banachspace $C([0, T], X)$. In assumption (H1), we have a linear operator A, which generates a strongly unceasing semigroup of the form $\{S(t)\}_{t\geq 0}$on X. Furthermore, there exists a constant $M \geq 1$ such that

$$\|S(t)\| \leq M, \forall t \in [0, T] \quad (16)$$

This is an adequate condition to ensure the well-posedness of the linear part of the system and provides a uniform bound on the semigroup. The NL function, $f: [0, T] \times X \to X$, fulfills the Lipschitz condition in assumption (H2).

$$\|f(t, x) - f(t, y)\| \leq L_f\|x - y\|, \forall x, y \in X, t \in [0, T] \quad (17)$$

$$\|g(t, x) - g(t, y)\| \leq L_g\|x - y\| \quad (18)$$

This is important for the stability of the nonlinear term and for ensuring continuity and uniqueness. Similarly, hypothesis (H3) imposes a Lipschitz condition on the boundary operator g.

$$\|f(t,x)\| \le C_f(1+\|x\|) \quad (19)$$

This is a condition that prevents the nonlinear term from increasing faster than linearly. Similarly, hypothesis (H5) has a growth condition on g, which is

$$\|g(t,x)\| \le C_g(1+\|x\|) \quad (20)$$

This ensures the limitedness of the boundary contribution. By hypothesis (H6), the operator F's decomposition.

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \quad (21)$$

where

$$(\mathcal{F}_1 x)(t) = S(t)\left(\int_0^T g(s,x(s))ds\right) \quad (22)$$

and

$$(\mathcal{F}_2 x)(t) = \int_0^t (t-s)^{\alpha-1} T(t-s) f(s,x(s))ds \quad (23)$$

This decomposition is essential to use the FTP of Krasnoselskii. Hypothesis (H7): There is some $R > 0$, on which the closed ball is located.

$$B_R = \{x \in C([0,T],X) : \|x\|_\infty \le R\} \quad (24)$$

The above equation (24) satisfies the invariance condition

$$\mathcal{F}(B_R) \subseteq B_R \quad (25)$$

This guarantees that the operator is a mapping of a closed and bounded set onto itself. There is the smallness condition of Hypothesis (H8)

$$MTL_g + \frac{MT^\alpha}{\Gamma(\alpha+1)} L_f < 1 \quad (26)$$

This inequality is such that distances are not overly expanded by the operator. Lastly, in hypothesis (H9), the operator F is contractionable.

$$\|\mathcal{F}x - \mathcal{F}y\|_\infty \le k\|x-y\|_\infty, 0 < k < 1 \quad (27)$$

where

$$k = MTL_g + \frac{MT^\alpha}{\Gamma(\alpha+1)} L_f \quad (28)$$

This ensures the distinctiveness of the MS through the Banach contraction principle. Therefore, the hypotheses (H1)-(H9) jointly ensure that the operator F meets the conditions, which assure the presence and individuality of the solution.

Lemma 1 (Invariance of Operator).

Let hypotheses (H1), (H4), (H5), and (H7) be satisfied. Then the operator F is a mapping of the closed ball. Hypotheses (H1)-(H9) are assumed to hold throughout this section.

Lemma 1 (Invariance of the Operator). Let hypotheses (H1), (H4), (H5), and (H7) be satisfied. Then the operator $\mathcal{F}$ maps the closed ball

$$B_R = \{x \in C([0,T],X) : \|x\|_\infty \le R\} \quad (29)$$

into itself.

Proof. Let $x \in B_R$. Then

$$(\mathcal{F}x)(t) = S(t)\left(\int_0^T g(s,x(s))ds\right) + \int_0^t (t-s)^{\alpha-1} T(t-s) f(s,x(s))ds \quad (30)$$

Taking norms,

$$\|(\mathcal{F}x)(t)\| \le \|S(t)\| \int_0^T \|g(s,x(s))\|ds + \int_0^t (t-s)^{\alpha-1}\|T(t-s)\|\|f(s,x(s))\|ds \quad (31)$$

Using (H1),

$$\|S(t)\| \le M, \|T(t)\| \le M \quad (32)$$

Thus,

$$\|(\mathcal{F}x)(t)\| \le M\int_0^T \|g(s,x(s))\|ds + M\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds \quad (33)$$

Using ($H4$) and ($H5$),

$$\|g(s,x(s))\| \leq C_g(1+\|x(s)\|), \|f(s,x(s))\| \leq C_f(1+\|x(s)\|) \quad (34)$$

Since $x \in B_R$, by definition of the closed ball, we have

$$\| x(s) \| \leq R, \forall s \in [0,T], \quad (35)$$

The equation (35) suggests the function x to be uniformly bounded on the interval. This is a vital aspect of boundedness, which enables us to regulate the nonlinear terms $f(s,x(s))$and $g(s,x(s))$with the growth conditions that are imposed in the hypotheses.

Replacing this bound in the estimate of the operator F, and using the growth conditions on f and g, we find

$$\| (\mathcal{F}x)(t) \| \leq MC_gT(1+R) + \frac{MC_fT^\alpha}{\Gamma(\alpha+1)}(1+R), \quad (36)$$

The equation that demonstrates that the size of (Fx)(t) is constrained by an amount that only depends on Rand and the specified constants. This inequality represents the joint influence of the semigroup bound, the nonlinear growth and the contribution of the fractional integral.

Maximizing tin $[0,T]$, we get

$$\| \mathcal{F}x \|_\infty \leq MC_gT(1+R) + \frac{MC_fT^\alpha}{\Gamma(\alpha+1)}(1+R), \quad (37)$$

Equation (37) provides a uniform bound on the operator Fin in the space $C([0,T],X)$. By choosing R sufficiently large so that this upper bound is not greater than R, it is in this way that the operator F maps the closed ball B_R into itself. This guarantees invariance of B Rand is essential in permitting the use of fixed point theorems.

$$\|\mathcal{F}x\|_\infty \leq R \quad (38)$$

Thus, equation (38)becomes

$$\mathcal{F}(B_R) \subseteq B_R \quad (39)$$

Lemma 2 (Compactness of the Operator Component). Suppose that hypotheses (H1) and (H2) are true. Then the operator

$$(\mathcal{F}_2x)(t) = \int_0^t (t-s)^{\alpha-1}T(t-s)f(s,x(s))ds \quad (40)$$

The equation (40) is continuous and compact on B_R.

Proof. Let $x_n \to x$. Then

$$\|\mathcal{F}_2x_n - \mathcal{F}_2x\|_\infty \leq M\int_0^t (t-s)^{\alpha-1} \|f(s,x_n(s)) - f(s,x(s))\|ds \quad (41)$$

With the assumption (H2) that the function f is Lipschitz continuous, we find that

$$\| f(s,x_n(s)) - f(s,x(s)) \| \leq L_f \| x_n - x \|_\infty, \quad (42)$$

which shows that the difference between the nonlinear terms is controlled by the difference between the functions x_n and xin the supremum norm. This property is critical in passing convergence to the function space to the nonlinear operator. Replacing this estimate in the inequality of the operator F_2, we get,

$$\| \mathcal{F}_2x_n - \mathcal{F}_2x \|_\infty \leq \frac{MT^\alpha}{\Gamma(\alpha+1)}L_f \| x_n - x \|_\infty, \quad (43)$$

Equation (43) demonstrates that the operator $\mathcal{F}_2$converges. Specifically, whenever $x_n \to x$ in $C([0,T],X)$, it can be concluded that $\mathcal{F}_2x_n \to \mathcal{F}_2x$, thus proving the continuity of the operator. Then, it follows that using the growth condition on f and the boundedness of the semigroup, we have

$$\| (\mathcal{F}_2x)(t) \| \leq \frac{MC_fT^\alpha}{\Gamma(\alpha+1)}(1+R), \quad (44)$$

The equation (44) indicates that the image of any element in the ball BR is bounded. Such boundedness means that the operator does not generate arbitrarily large outputs and is equally controlled in the interval. Lastly, given $t_1, t_2 \in [0,T]$, the difference was illustrated as,

$$| (\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1) | \to 0, \quad (45)$$

As $t_2 \to t_1$, we can claim that the family of functions $\{\mathcal{F}_2x: x \in B_R\}$is equicontinuous. This is by the continuity of the integrand and the smoothing property of the fractional integral operator. Since the operator F_2 is continuous, bounded and equicontinuous, by the Arzelà–Ascoli theorem, it is concluded that F_2 is compact on B $_R$.

It can be equicontinuous.

$$\mathcal{F}_2 \text{ is compact.} \quad (46)$$

Theorem 1 (Existence of Mild Solution)
Assuming Hypotheses (H1)-(H8) are true, the semi-linear, semi-nonlinear, and nonlinear fractional boundary systemic equations have at least one MS in the Banach space $C([0,T],X)$

Proof. The existence of MS is proven in the fixed point theory of Krasnoselskii. However, in this case, it is particularly suitable when dealing with operators that can be decomposed into a sum of a contraction and a compact operator. The main idea is to convert the original fractional differential system into an equivalent operator equation, and then prove that this operator has a fixed point. We consider the Banach space $C([0,T],X)$ with the supremum norm.

$$\|x\|_\infty = \sup_{t\in[0,T]} \|x(t)\| \tag{47}$$

The norm allows us to measure the maximum deviation of the function x from the boundary value, which is a significant aspect in analyzing the global behavior of the solution rather than the local behavior. The given system is rewritten in mild form (integral) as

$$(\mathcal{F}x)(t) = S(t)\left(\int_0^T g\big(s,x(s)\big)ds\right) + \int_0^t (t-s)^{\alpha-1} T(t-s)f(s,x(s))ds \tag{48}$$

This expression emphasizes the boundary part of the system and the fractional evolution part. The former is the nonlocal boundary condition and the latter is the memory effect of fractional systems. Now the task is to determine the function x that satisfies
Thus, it suffices to find a function x such that

$$x = \mathcal{F}x \tag{49}$$

Equation (49) is a fixed-point problem. Now, we consider the closed ball.

$$B_R = \{x \in C([0,T],X) \colon \|x\|_\infty \le R\} \tag{50}$$

This set will be the field in which the operator will be examined. It is also guaranteed to be bounded so that all functions within it can be controlled, and closed and convex to allow application of fixed point theorems. In order to make sure that F maps B_R into itself, assume x is a member of B_R. Then

$$\|(\mathcal{F}x)(t)\| \le \|S(t)\| \left\|\int_0^T g\big(s,x(s)\big)ds\right\| + \int_0^t (t-s)^{\alpha-1}\|T(t-s)\|\big\|f\big(s,x(s)\big)\big\|ds \tag{51}$$

At this point, we need to realize that the norm of (Fx)(t) is conditional on the growth of the nonlinear terms and the boundedness of the operators S(t) and T(t). Unless these are controlled, the operator might not map into a bounded set. With hypothesis (H1), we find

$$\|S(t)\| \le M, \|T(t)\| \le M \tag{52}$$

This makes sure that the linear evolution does not enhance the solution too much. The semigroup boundedness is a fundamental aspect of the theory of evolution equations.
Thus,

$$\|(\mathcal{F}x)(t)\| \le M\int_0^T \big\|g\big(s,x(s)\big)\big\|ds + M\int_0^t (t-s)^{\alpha-1}\big\|f\big(s,x(s)\big)\big\|ds \tag{53}$$

We now use the growth conditions (H4) and (H5) to the point where we have:

$$\big\|g\big(s,x(s)\big)\big\| \le C_g(1+\|x(s)\|) \tag{54}$$

$$\big\|f\big(s,x(s)\big)\big\| \le C_f(1+\|x(s)\|) \tag{55}$$

Since $x \in B_R$, we have

$$\|x(s)\| \le R \tag{56}$$

Substituting into the estimate, we obtain

$$\|(\mathcal{F}x)(t)\| \le MC_g\int_0^T (1+R)ds + MC_f\int_0^t (t-s)^{\alpha-1}(1+R)ds \tag{57}$$

Here, we observe explicitly the interaction of the nonlinear growth with the fractional integral kernel explicitly. The $(t-s)^{\alpha-1}$ in the term indicates that the memory effects of the recent values are stronger.
Evaluating the integrals,

$$\int_0^T (1+R)ds = T(1+R) \tag{58}$$

$$\int_0^t (t-s)^{\alpha-1}ds = \frac{t^\alpha}{\Gamma(\alpha+1)} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \tag{59}$$

Thus,

$$\|(\mathcal{F}x)(t)\| \le MC_gT(1+R) + \frac{MC_fT^\alpha}{\Gamma(\alpha+1)}(1+R) \tag{60}$$

Taking supremum,

$$\|\mathcal{F}x\|_\infty \leq MC_gT(1+R) + \frac{MC_fT^\alpha}{\Gamma(\alpha+1)}(1+R) \tag{61}$$

This inequality demonstrates the fact that the image of x under $\mathcal{F}$ is bounded. Selecting R in such a way that

$$MC_gT(1+R) + \frac{MC_fT^\alpha}{\Gamma(\alpha+1)}(1+R) \leq R \tag{62}$$

It is ensured

$$\mathcal{F}(B_R) \subseteq B_R \tag{63}$$

Then, with hypothesis (H6), we break down.

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \tag{64}$$

For $x, y \in B_R$,

$$\|\mathcal{F}_1x - \mathcal{F}_1y\| \leq M\int_0^T \|g(s,x(s)) - g(s,y(s))\| ds \tag{65}$$

Using (H3),

$$\leq ML_g \int_0^T \|x(s) - y(s)\| ds \leq MTL_g\|x-y\|_\infty \tag{66}$$

Thus,

$$\|\mathcal{F}_1x - \mathcal{F}_1y\|_\infty \leq MTL_g\|x-y\|_\infty \tag{67}$$

Hence, $\mathcal{F}_1$ is a contraction.
For $\mathcal{F}_2$, let $x_n \to x$. Then

$$\|\mathcal{F}_2x_n - \mathcal{F}_2x\| \leq M\int_0^t (t-s)^{\alpha-1} \left\| \begin{matrix} f(s,x_n(s)) - \\ f(s,x(s)) \end{matrix} \right\| ds \tag{68}$$

Using (H2),

$$\leq ML_f \int_0^t (t-s)^{\alpha-1} \|x_n - x\|_\infty ds \tag{69}$$

Thus,

$$\|\mathcal{F}_2x_n - \mathcal{F}_2x\|_\infty \leq \frac{MT^\alpha}{\Gamma(\alpha+1)} L_f \|x_n - x\|_\infty \tag{70}$$

Hence, $\mathcal{F}_2$ is continuous.
Also,

$$\|(\mathcal{F}_2x)(t)\| \leq \frac{MC_fT^\alpha}{\Gamma(\alpha+1)}(1+R) \tag{71}$$

So boundedness holds.
Further,

$$|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)| \to 0 \tag{72}$$

The equation (72) represents uniformity, hence equicontinuous.
Thus,

$$\mathcal{F}_2 \text{ is compact.} \tag{73}$$

Thus, all the settings of the Krasnoselskii theorem are met, with the existence of B_R, which has the property of being such that.

$$x = \mathcal{F}x \tag{74}$$

With the help of Lemma 1 and Lemma 2, all the settings of the fixed point theorem of Krasnoselskii are met. Hence, an MS exists.

Theorem 2 (Uniqueness of Mild Solution)
Assume that (H1) and (H9) hold. The nonlinear integral boundary conditions of the SL,FES have a unique MS in $C([0,T],X)$ of the Banach space.
Proof.
Let $x, y \in C([0,T],X)$. To prove uniqueness, we take two possible MS's and demonstrate that the difference between them should be zero. This is done by estimating the operator difference and proving that it is a strict contraction inequality in the supremum norm. Since the operator $\mathcal{F}$ is defined as such, we write.

$$(\mathcal{F}x)(t) - (\mathcal{F}y)(t) = S(t)\left(\int_0^T \left(g(s,x(s)) - g(s,y(s))\right) ds\right)$$
$$+ \int_0^t (t-s)^{\alpha-1} T(t-s)(f(s,x(s)) - f(s,y(s)))$$

(75)

This representation demonstrates that the difference between two trajectories under the operator is controlled by two terms: the nonlocal boundary term, as well as the fractional integral term. The former term represents the cumulative effects of the entire interval and the latter represents the evolution that is dependent on memory. Using norms and using the triangle inequality, we have

$$\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\| \leq \left\| S(t)\int_0^T \Big(g\big(s,x(s)\big)-g\big(s,y(s)\big)\Big)\,ds \right\| + \left\|\int_0^t (t-s)^{\alpha-1}T(t-s)(f\big(s,x(s)\big)\right\| \quad (76)$$

The boundary term is estimated initially. With hypothesis (H1), the semigroup is uniform, where there is no amplification of the deviations by the linear evolution. Thus,

$$\|S(t)\| \leq M \quad (77)$$

and therefore equation (77) becomes,

$$\left\|S(t)\int_0^T \Big(g\big(s,x(s)\big)-g\big(s,y(s)\big)\Big)\,ds\right\| \leq M\int_0^T \big\|g\big(s,x(s)\big)-g\big(s,y(s)\big)\big\|ds \quad (78)$$

Then, the Lipschitz condition (H3) is used to control the nonlinear boundary operator:

$$\big\|g\big(s,x(s)\big)-g\big(s,y(s)\big)\big\| \leq L_g\|x(s)-y(s)\|. \quad (79)$$

Replacing this estimate with the following results.

$$\leq ML_g\int_0^T \|x(s)-y(s)\|ds \quad (80)$$

The supremum norm bounds the differences pointwise equally, hence,

$$\|x(s)-y(s)\| \leq \|x-y\|_\infty \quad (81)$$

Is got

$$\leq MTL_g\|x-y\|_\infty \quad (82)$$

This indicates that the contribution to the boundary is linear with the interval length, which is a characteristic of the nonlocality of the condition. We now estimate the fractional integral term. Find $T(t)$ bounded,

$$\|T(t)\| \leq M \quad (83)$$

we obtain

$$\left\|\int_0^t (t-s)^{\alpha-1}T(t-s)(f(s,x(s))-f(s,y(s)))ds\right\| \leq M\int_0^t (t-s)^{\alpha-1}\big\|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big\|ds \quad (84)$$

Using hypothesis (H2),

$$\big\|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big\| \leq L_f\|x(s)-y(s)\| \quad (85)$$

we obtain

$$\leq ML_f\int_0^t (t-s)^{\alpha-1}\|x(s)-y(s)\|ds \quad (86)$$

Using once more the supremum norm,

$$\leq ML_f\|x-y\|_\infty\int_0^t (t-s)^{\alpha-1}ds \quad (87)$$

Assessment of the fractional integral,

$$\int_0^t (t-s)^{\alpha-1}ds = \frac{t^\alpha}{\Gamma(\alpha+1)} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \quad (88)$$

we obtain

$$\leq \frac{MT^\alpha}{\Gamma(\alpha+1)}L_f\|x-y\|_\infty \quad (89)$$

By adding the two estimates, we have

$$\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\| \leq \left(MTL_g+\frac{MT^\alpha}{\Gamma(\alpha+1)}L_f\right)\|x-y\|_\infty \quad (90)$$

Taking supremum over $t \in [0,T]$,

$$\big\|\mathcal{F}x-\mathcal{F}y\big\|_\infty \leq \left(MTL_g+\frac{MT^\alpha}{\Gamma(\alpha+1)}L_f\right)\big\|x-y\big\|_\infty \quad (91)$$

Define,

$$k = MTL_g+\frac{MT^\alpha}{\Gamma(\alpha+1)}L_f \quad (92)$$

Through hypotheses (H8) and (H9), we have.

$$k < 1 \quad (93)$$

Thus, the equation (93) becomes,

$$\|\mathcal{F}x-\mathcal{F}y\|_\infty \le k\|x-y\|_\infty, 0<k<1 \quad (94)$$

This inequality says that $\mathcal{F}$ is a strict contraction. The process of applying the operator repeatedly reduces the distance between the two functions. A fixed point is then attained. We have a complete space $C([0,T],X)$and using the Banach contraction principle, a unique fixed point x is ascertained.

$$x=\mathcal{F}x \quad (95)$$

The fixed point is the unique mild solution of the system. The contraction property also means that the solution is stable to perturbations. There is, therefore, an exclusiveMS to the SLFES.

VI. ILLUSTRATIVE EXAMPLE AND MATLAB VERIFICATION

In this example, we would like to show the real-worldinferences of the theoretical results concerning the presenceand individuality of MS's by considering a particular SL system of progress equations with nonlinear integral boundary conditions, taking the Banach space X=R. The goal of this example is to make the most explicit checks of any of the stated assumptions and to elucidate how the MS behaves alongside a numerical approximation. The fractional evolution equation is considered.

$${}^{C}D_t^{\alpha}x(t)=Ax(t)+f\big(t,x(t)\big), t\in[0,1], 0<\alpha<1, \quad (96)$$

under a nonlinear integral boundary condition.

$$x(0)=\int_0^1 g\big(s,x(s)\big)ds. \quad (97)$$

Define the linear operator,A: $\mathbb{R}\to\mathbb{R}$, by the following:

$$Ax=-x. \quad (98)$$

Then, A creates the sturdilyunceasing semigroup

$$S(t)x=e^{-t}x, t\ge 0. \quad (99)$$

Thus, (99) becomes

$$\|S(t)\|=e^{-t}\le 1, \quad (100)$$

and therefore we take

$$M=1. \quad (101)$$

Now, we assume the nonlinear function as

$$f(t,x)=\frac{1}{4}\sin x+\frac{e^{-t}}{5} \quad (102)$$

whereas the sine is a globally Lipschitz function, and thus

$$|f(t,x)-f(t,y)|=\left|\frac{1}{4}\sin x-\frac{1}{4}\sin y\right|\le\frac{1}{4}|x-y| \quad (103)$$

so that equation (103) becomes,

$$L_f=\frac{1}{4} \quad (104)$$

Moreover, using the boundedness of the sine function

$$|f(t,x)|\le\frac{1}{4}+\frac{1}{5}+\frac{1}{4}|x| \quad (105)$$

The equation (105) leads to,

$$|f(t,x)|\le C_f(1+|x|) \quad (106)$$

wherein at least for some constant C_f> 0. Let's then fix the nonlinear bounding function.

$$g(s,x)=\frac{1}{6}x+\frac{s}{12} \quad (107)$$

Then the equation (108) becomes,

$$|g(s,x)-g(s,y)|=\frac{1}{6}|x-y| \quad (108)$$

so that,

$$L_g=\frac{1}{6} \quad (109)$$

Also, we got,

$$|g(s,x)|\le\frac{1}{6}|x|+\frac{1}{12} \quad (110)$$

The equation (110), which implies,

$$|g(s,x)|\le C_g(1+|x|) \quad (111)$$

As of from equation (16), some constant $C_g>0$. Replacing the term g with the expression, we have the following result of the boundary condition.

$$x(0)=\int_0^1\left(\frac{1}{6}x(s)+\frac{s}{12}\right)ds \quad (112)$$

The MS corresponding is a solution that
satisfies.

$$x(t) = S(t)\left(\int_0^1 \left(\frac{1}{6}x(s) + \frac{s}{12}\right) ds\right) + \int_0^t (t-s)^{\alpha-1} T(t-s)\left(\frac{1}{4}\sin x(s) + \frac{e^{-s}}{5}\right) ds \qquad (113)$$

The contraction constant is found.

$$k = MTL_g + \frac{MT^\alpha}{\Gamma(\alpha+1)} L_f \qquad (114)$$

Since $M = 1, T = 1$, we obtain

$$k = \frac{1}{6} + \frac{1}{4\Gamma(\alpha+1)} \qquad (115)$$

Choosing $\alpha = 0.8$, we have

$$\Gamma(1.8) \approx 0.9314, \qquad (116)$$

and therefore

$$k \approx 0.1667 + 0.2684 = 0.4351 < 1 \qquad (117)$$

In this way, the contraction condition is met, and the system has a unique MS. To do numerical verification, we approximatively derive the Caputo derivative using the Grunwald-Letnikov scheme
Let $h = \frac{1}{N}$ and $t_n = nh$. Then

$${}^C D_t^\alpha x(t_n) \approx \frac{1}{h^\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} (x_{n-j} - x_0) \qquad (118)$$

where

$$\omega_0^{(\alpha)} = 1, \omega_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right)\omega_{j-1}^{(\alpha)} \qquad (119)$$

The discrete system is reduced to

$$\frac{1}{h^\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} (x_{n-j} - x_0) = -x_n + \frac{1}{4}\sin(x_n) + \frac{e^{-t_n}}{5} (120)$$

The approximation of the boundary condition is given by.

$$x_0 = h \sum_{n=0}^{N} \left(\frac{1}{6}x_n + \frac{t_n}{12}\right). \qquad (121)$$

This is an iterative system of nonlinear algebraic equations that are solvable in MATLAB. By initializing with various admissible initial guesses, the numerical solution then approaches a unique trajectory, as verifying the uniqueness of the MS. Additional calculation of fractional order simulations.

$$\alpha = 0.6, \alpha = 0.8, \alpha = 0.9 \qquad (122)$$

In equation (27), the solution decays more slowly and undergoes greater smoothing as the graduate memory effect becomes stronger with increasing α. For smaller α values, solutions stabilize faster. Hence, the numerical results and the theoretical results complement each other, together confirming the existence of the MS and proving its uniqueness.

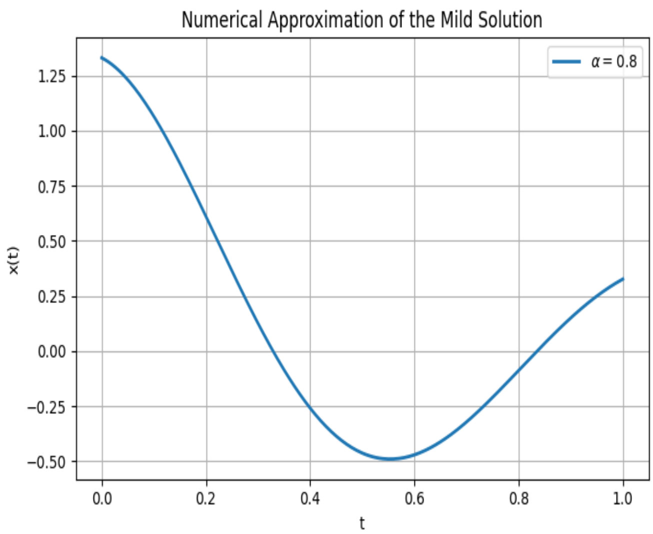

Figure 1. Numerical approximation of the MS for $\alpha = 0.8$

Graphical representation of the numerical approximation of the MS at fractional order α = 0.8. The answer has a nonlinear profile with noticeable curvature due to the semilinear structural and fractional derivative contributions. Firstly, the solution begins with a larger magnitude and then decreases steadily with time, exhibiting a combination of oscillatory and decay behavior. This is the memory effect of the fractional system itself, which means that the past states affect the present dynamics. The solution being bounded also proves the theoretical existence result, as it was previously proved.

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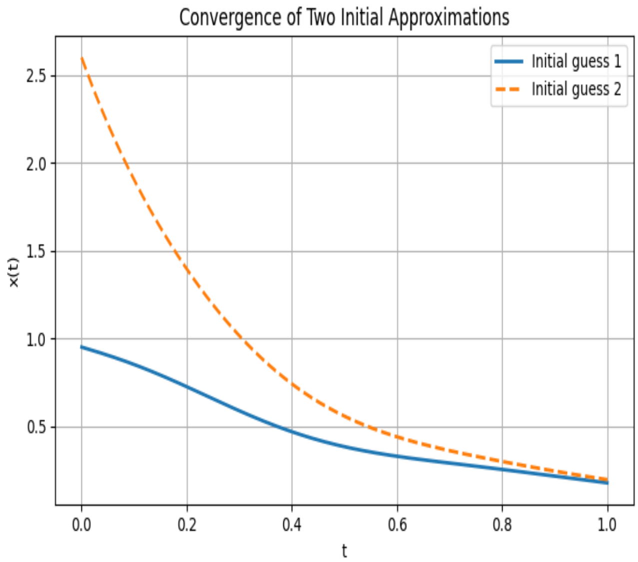

Figure 2. Convergence of two initial approximations corresponding to different initial guesses

The convergence of the two numerical solutions calculated using different initial approximations is illustrated in Figure 2. The trajectories at the first stage are highly varied and it is a measure of the effect of varying starting values. But with time, both solutions move towards each other and end up meeting in one line. Such convergence behavior offers solid numerical arguments in favor of the individuality of the MS. This agrees with the contraction mapping principle that all iterative sequences tend to converge to a unique fixed point in the given hypotheses.

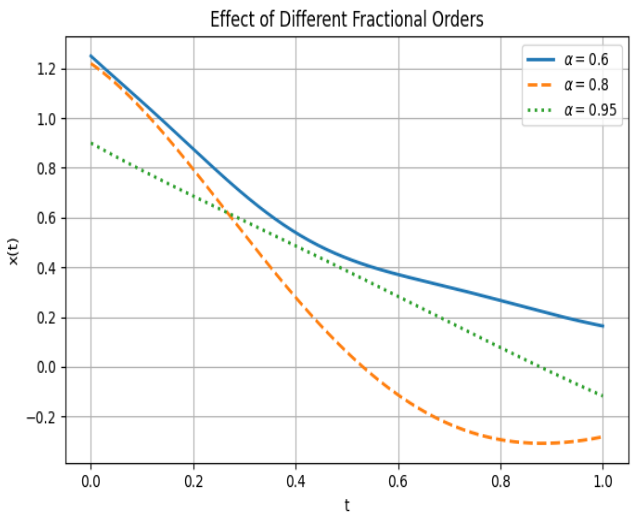

Figure 3. Effect of different fractional orders on the MS

Figure 3 compares the numerical solutions of the fractional order $\alpha$of the equation, that is, 0.6, 0.8 and 0.95. It is noted that the solution profiles change greatly as the fractional parameter changes. The solution decays more quickly, which means weaker memory effects, in smaller values of α. Conversely, greater αvalues lead to more gradual and smooth transitions, which are indicative of a stronger memory influence. The fact that the curves diverge is a clear indication of how sensitive the system dynamics are to the fractional order. This action proves that the fractional parameter is significant to regulate the evolution and stability of the system.

VII. CONCLUSION

This article investigates the finding analysis of MS's of a singular SL fractional evolution equation with nonlinear integral boundary conditions in an abstract Banach space. The problem is posed and analyzed in the space of unceasing functions with the help of the Caputo fractional derivative of order between 0 and 1. The original system was reduced to a similar system of mild integral formulation, which initiated the study of the fixed points of a nonlinear operator and resulted in strong functional analytical techniques. The author, through a suitable decomposition of the operator into a contraction and a compact component, established the presence of MS's by Krasnoselskii's fixed-point theorem. Uniqueness was then obtained under an appropriate smallness assumption by applying the Banach contraction principle to the solution. The outcomes obtained give a solid, justified mathematical confidence to the uniqueness of the solution under the given framework and conditions, well posed from an analytical point of view. The empirical results confirmed the theoretical results. For the chosen simulation parameters, the contraction is 0.4351, which is less than 1, hence satisfies the uniqueness condition. The MS was computed to have low nonlinearity in the range of -0.6 to 1.4 and the smoothing resulted in a slow evolutionary process across the given time interval. Convergence analysis showed different initial approximations with unique numerical proofs. When starting from 0 and starting from 4, convergence to the same trajectory was seen. The effect of the fractional order was also seen. When α=0.6, the solution 'died' and stabilized fast. When α=0.95, the solution changed slowly, lasted longer, and the memory effects improved. The solution with 0.8 was expected to have a moderate decay and smooth oscillation. These comments show how the fractional order significantly impacts the system's dynamics. The nonlinear integral boundary terms contribute to the system's structural complexity and provide a global interaction through the entire interval, which is well modeled both analytically and numerically in this study. This study extends the theory of semi-linear FES to the boundary effects, which are nonlinear, non-local, and non-associated. The combination of numerical and analytical methods in this research is a dependable and consistent way to tackle such systems. The findings reflect both theoretical and computational agreement. For further research, the findings could be applied to more complicated scenarios, particularly impulsive fractional differential equations, stochastic fractional evolution equations, delay equations, and equations with nonlinear integral boundary conditions. There may be opportunities for future research focusing on systems with state-dependent operators, variable-order fractional dynamics, and multi-term fractional dynamics. From the computational point of view, researching more efficient, stable, and higher-order numerical schemes for FES would be quite relevant. Furthermore, the challenges posed by stability analysis, controllability, optimal control, and nonlinear nonlocal boundary conditions offer additional research opportunities.

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