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Fuzzy Soft Set Approach to First-Order Separable Differential Equations

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Abstract

This paper investigates first-order separable differential equations in a fuzzy soft set environment. The concept of fuzzy soft sets combines the uncertainty-handling capability of fuzzy sets with the parameter-ization feature of soft sets, making it suitable for modeling systems involving multiple sources of uncertainty. In this work, fuzzy soft differential equations with parameter-dependent coefficients are considered, and solution techniques are developed by transforming them into corresponding crisp differential equations for each parameter. The study discusses Type-1, Type-2 and Type-3 fuzzy soft differential equations and analytical solutions are obtained using classical methods such as separation of variables and integrating factor techniques. The obtained results show that fuzzy soft differential equations generate families of solutions associated with different parameters, thereby providing a richer and more flexible representation of uncertainty in dynamical systems.

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Introduction

The study of uncertainty has become an important area in applied mathematics, engineering and com-putational sciences. Classical mathematical models often fail to describe systems involving imprecise, incomplete, or vague information. To address this limitation, Zadeh introduced the concept of fuzzy sets, which allows elements to have degrees of membership ranging between 0 and 1. Molodtsov introduced soft set theory as a mathematical tool for dealing with uncertainty without requiring additional assumptions such as membership grades or probability distributions. The combination of these two frameworks led to the development of fuzzy soft set theory, which provides a more flexible structure for handling uncertainty in complex systems. Fuzzy differential equations extend classical differential equations by incorporating fuzzy-valued functions and have been widely studied using Hukuhara and generalized Hukuhara derivatives. However, in many real-world problems, uncertainty arises not only in state variables but also in parameters governing the system. This motivates the use of fuzzy soft differential equations, where parameters them-selves are represented as fuzzy soft sets. In this paper, analyze first-order separable fuzzy soft differential equations under different structural forms. The proposed framework converts fuzzy soft systems into para-metric crisp differential equations, enabling analytical solutions using classical techniques. Three types of fuzzy soft differential equations are discussed, illustrating how fuzzy soft coefficients influence the solution structure. The results show that fuzzy soft differential equations generate a family of solutions corresponding to each parameter in the soft set, thereby providing a more comprehensive representation of uncertainty in dynamical systems. This approach is useful in engineering, physics and applied sciences where parameters are inherently uncertain and multi-valued. The rest of the paper is organized as follows. Preliminary concepts of fuzzy sets and fuzzy soft sets are introduced first, followed by the formulation of fuzzy soft differential equations. Solution methodologies are then presented along with illustrative examples the study concludes with a discussion of key findings and future research directions.

Fuzzy Set. In 1965, an American mathematician and computer scientist Lotfi A. Zadeh [19] introduced the fuzzy set concept. In the fuzzy set, there are ordered pair set elements instead of single elements, where the first is the element itself followed by its membership degree or degree of belongingness. The definitions and properties are described as follows. Consider U be a universal set and F as a fuzzy set defined on it. The fuzzy set$F$ is defined as $F= \{(x, \mu_F(x)): x\epsilon U\}$, where μ_F :U $\rightarrow$ [0, 1] is a membership function and $x\epsilon U$ is arbitrary element.

Fuzzy Arithmetic Operations. Let $\tilde{u}, \tilde{v} \in \Omega$ be fuzzy numbers and k be a scalar. Then the arithmetic operations between $u^{\sim}$ and $v^{\sim}$ are defined as follows.

Addition

$[\tilde{u}]^{\alpha} \oplus [\tilde{v}]^{\alpha} = [\underline{u}, \bar{u}] \oplus [\underline{v}, \bar{v}] = [\underline{u} + \underline{v}, \bar{u} + \bar{v}]$,

Scalar multiplication when $k > 0$.

$k \otimes [\tilde{u}]^{\alpha} = k[\underline{u}, \bar{u}] = [k\underline{u}, k\bar{u}] \quad \forall \alpha\epsilon(0,1]$.

When $k \leq 0$,

$k \otimes [\tilde{u}]^{\alpha} = k[\underline{u}, \bar{u}] = [k\bar{u}, k\underline{u}], \quad \forall \alpha\epsilon(0,1]$.

Multiplication

$[\tilde{u}]^{\alpha} \oplus [\tilde{v}]^{\alpha} = [\underline{u}, \bar{u}] \oplus [\underline{v}, \bar{v}] = [\underline{w}, \bar{w}]$,

Where

$\underline{w}$ =min $\{\underline{uv}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{\bar{u}}\bar{\bar{v}}\}$,

And

$\bar{w}$=max$\{\underline{uv}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{\bar{u}}\bar{\bar{v}}\}$.

α-cut of Fuzzy Set. The α-cut of a fuzzy set is a parametric representation of the fuzzy set where it is represented as a classical set. In α -cut of a fuzzy set, those elements whose membership values are greater than or equal to α are included.

α-cut: Let $H^{\sim}$ be a fuzzy set with membership function μ_H defined on a universal set U. Then $\mu_{\alpha} = \{x: \mu_H(x) \geq \alpha, x\epsilon U\}$, is called the α-cut of the fuzzy set H, where $\alpha \in$ [0, 1].

Strong α-cut: Let H be a fuzzy set with membership function μ_H defined on a universal set U. Then $\mu_{\alpha} = \{x: \mu_H(x) > \alpha, x\epsilon U\}$, is called the strong α −cut the fuzzy set H, where $\alpha \in$ [0, 1].

Fuzzy Number. A fuzzy number [12] is a generalisation of a regular real number in that it refers to a connected collection of alternative values rather than a single one, and each possible value has a weight ranging from 0 to 1. The membership function is the name given to this weight. On the other hand, a convex, normalised fuzzy set on the real line is a particular

instance of a fuzzy number.
A fuzzy set $E = \{\tilde{u} : R \rightarrow [0, 1]\}$ is said to be a fuzzy number if it satisfies the following properties
(1) $\tilde{u}$ is normal that is the membership value should be 1 for at least one point.
(2) The set $\tilde{u}$ is a convex fuzzy set $\mu_{\tilde{u}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{u}}(x_1), \mu_{\tilde{u}}(x_2)\}$, for all $x_1, x_2 \epsilon \tilde{u}$ and $0 \leq \lambda \leq 1$.
(3) The 0-cut of a fuzzy set

$$0_{\tilde{u}=(x\, \epsilon X : \mu_{\tilde{u}}(x) \geq 0\}}$$

should be bounded.
(4) The membership function of the fuzzy set must be piecewise continuous.

Definition. A triangular fuzzy number $\tilde{a}$ is written as
$\tilde{a} = (a_1, a_2, a_3)$
which is a convex normalized fuzzy set on the real line $\mathbb{R}$ such that
- There exists a unique $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{a}}(x_0) = 1$, where x_0 is called the mean value of $\tilde{a}$ and $\mu_{\tilde{a}}$ the membership function.
- $\mu_{\tilde{a}}(x)$ is piecewise continuous.

The membership function is defined as

$$\begin{cases} 0, & x < a_1, \\ \dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ 0, & x \geq a_3. \end{cases}$$

Definition: Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be two non-negative negative triangular fuzzy numbers where $a_1 \geq 0$ and $b_1 \geq 0$. Let k be a real number. Then operations are defined as

$$k\tilde{a} = \begin{cases} (ka_1, ka_2, ka_3), & k \geq 0, \\ (ka_3, ka_2, ka_1) & k < 0, \end{cases}$$

$$\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3),$$

$$\tilde{a} - \tilde{b} = (a_1 - b_3, a_2 - b_2, a_3 - b_1),$$

Fuzzy Soft Set. Let U be a universal set and E be a set of parameters. A fuzzy soft set over U is a pair (F, E), where F is a mapping defined as

$$F: E \rightarrow \mathcal{F}(U)$$

such that for each parameter $e \epsilon E$, $F(e)$ is a fuzzy set on U. That is, $F(e) = \{(x, \mu_{F(e)}(x)) : x \epsilon U, \mu_{F(e)}(x) \epsilon [0,1]$.

Definition (Fuzzy Soft Set Coefficients). Let X be a universe of discourse and E be a set of parameters. A fuzzy soft coefficient is a parameterized fuzzy value represented by a fuzzy soft set (F, E), where for each parameter $e \in E$, the mapping
$F(e) : X \rightarrow [0, 1]$
assigns a membership value to every element of X. In fuzzy soft differential equations, these coefficients are used to describe uncertainty in the parameters of the differential equation. For a parameter $e \in E$, the fuzzy soft coefficient produces a corresponding crisp coefficient, leading to a family of differential equations associated with different parameters. Example, if
$(F,E) = \{e_1 \rightarrow 2,\ e_2 \rightarrow 3\}$,
then the coefficient takes the value 2 for parameter e_1 and 3 for parameter e_2.

Fuzzy Soft Set Operations. Let $U = \{x_1, x_2\}$ be a universe of discourse and $E = \{e_1, e_2\}$ be the parameter set. Define two fuzzy soft sets(F,E) and (G,E)

Fuzzy Soft Set *F*

$$F(e_1) = \{x_1 : (1, 2, 3),\ x_2 : (2, 3, 4)\}$$
$$F(e_2) = \{x_1 : (2, 3, 5),\ x_2 : (1, 2, 3)\}$$

Fuzzy Soft Set *G*

$$G(e_1) = \{x_1 : (0, 1, 2),\ x_2 : (1, 2, 3)\}$$
$$G(e_2) = \{x_1 : (1, 1.5, 2),\ x_2 : (2, 2.5, 3)\}$$

1. **Union of Fuzzy Soft Sets.** Using triangular fuzzy addition

$$\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3),$$

For e_1:
x_1 : (1, 2, 3) + (0, 1, 2) = (1, 3, 5)
x_2 : (2, 3, 4) + (1, 2, 3) = (3, 5, 7)

$(F \cup G)(e_1) = x_1 : (1, 3, 5),\ x_2 : (3, 5, 7)\}$

For e_2:
x_1 : (2, 3, 5) + (1, 1.5, 2) = (3, 4.5, 7)
x_2 : (1, 2, 3) + (2, 2.5, 3) = (3, 4.5, 6)
$(F \cup G)(e_2) = \{x_1 : (3, 4.5, 7),\ x_2 : (3, 4.5, 6)\}$

2. Intersection of Fuzzy Soft sets:

$$\tilde{a} \cap \tilde{b} = (\min(a_1, b_1), \min(a_2, b_2), \min(a_3, b_3))$$

For e_1

$$x_1: (1,2,3) \cap (0,1,2) = (0,1,2)$$
$$x_2: (2,3,4) \cap (1,2,3) = (1,2,3)$$
$$(F \cap G)(e_1) = \{x_1: (0,1,2), x_2: (1,2,3)\}$$

For e_2

$$x_1: (2,3,5) \cap (1,1.5,2) = (11.5,,2)$$
$$x_2: (1,2,3) \cap (2,,2.5,3) = (1,2,3)$$
$$(F \cap G)(e_2) = \{x_1: (1,1.5,2), x_2: (1,2,3)\}$$

3. Scalar Multiplication Example: Let k=2,

$$k\tilde{a} = (2a_1, 2a_2, 2a_3)$$

For $F(e_1)$:
$x_1: (1,2,3) \rightarrow (2,4,6)$
$x_2: (2,3,4) \rightarrow (4,6,8)$

Fuzzy Soft Set Approach to Fuzzy Differential Equation.

Fuzzy Soft Differential Equation. Let U be a universal set and E be a set of parameters. A fuzzy soft set is denoted by (F, E), where for each $e \in E$, $F(e, t)$ is a fuzzy-valued function of $t \in \mathbb{R}$. A fuzzy soft differential equation is an equation of the form

$$\frac{d}{dt}F(e,t) = G(e,t,F((e,t))),\quad e\epsilon\ E, t\epsilon I$$

Where

- F(e,t) is a fuzzy- values function,
- G is a fuzzy soft mapping,
- $I \subset \mathbb{R}$ is an Interval,
- The derivative is understood in the sense of Hukuhara derivative or generalized Hukuhara (gH)derivative.

α-cut Representation. Using α-cut representation, the fuzzy soft differential equation becomes

$$\frac{d}{dt}[F(e,t)]^{\alpha} = [G(e,t,F((e,t))]^{\alpha}$$
$$[F(e,t)]^{\alpha} = [\underline{F}_{\alpha}(e,t), F_{\alpha}(e,t)]$$

And $[\underline{F}_{\alpha}(e,t), F_{\alpha}(e,t)]$ be lower bound of the α −cut and upper bound of the α −cut. So the FSDE reduces to a system of interval differential equations:

$$\frac{d}{dt}\underline{F}_{\alpha}(e,t) = \underline{G}_{\alpha}(e,t)$$
$$\frac{d}{dt}\overline{F}_{\alpha}(e,t) = \overline{G}_{\alpha}(e,t)$$

Initial Value Problem (Fuzzy Soft IVP):

$$\frac{d}{dt}F(e,t) = G(e,t,F(e,t)), F(e,t_0) = F_0(e)$$

Where $F_0(e)$ is a fuzzy soft initial condition.

Example (Simple FSDE). Consider

$$\frac{d}{dt}F(e,t) = -\lambda F(e,t),\quad \lambda > 0$$

Solution:

$$F(e,t) = F_0(e)(e)^{-\lambda t}, e\epsilon E$$

Step 1: Define parameter set

$$A = \{e_1, e_2\}$$

Assume fuzzy coefficients

$$\tilde{a}(e_1) = 1,\ \tilde{a}(e_2) = 2$$

Step 2: Convert into crisp differential equations

Case 1: $e = e_1\ \frac{dx}{dt} = x$

Case 2: $e = e_2 \frac{dx}{dt} = 2x$

Step 3: Solve each equation

Case 1: $\frac{1}{x} dx = dt$

$In\, x = t + C$

$x_1(t) = C_1 e^t$

Case 2: $\frac{1}{x} dx = 2dt$

$In\, x = 2t + C$

$x_1(t) = C_2 e2^t$

Step 4: Fuzzy soft set solution

$$x_F(t,e) = \begin{cases} C_1 e^t, e = e_1 \\ C_2 e^{2t}, e = e_2 \end{cases}$$

$$\frac{dx_F}{dt} = \tilde{a}(e) x_F \Rightarrow x_F(t,e) = C(e) e^{\tilde{a}(e)t}, \quad e \epsilon A$$

First-Order separable differential equation with fuzzy soft set coefficient Let X be the universe ofdiscourse and E be a set of parameters. Suppose (F, E) and (G , E) are fuzzy soft sets representing uncertain coefficients of a differential equation. A differential equation of the form

(F,E)$\phi(y)dy$ =(G,E) ψ(x) dx

Or equivalently

$$\frac{dy}{dx} = \frac{(\mathrm{G,E})\ \psi(\mathrm{x})}{(\mathrm{F,E})\phi(y)}$$

is called a **first-order separable differential equation with fuzzy soft set coefficients** if the variables x and y can be separated into different sides of the equation and the coefficients are described by fuzzy soft sets.

- $\phi(y)$ is a function only of y,
- $\psi(x)$ is a function only of x,
- (F, E) and(G,E) are fuzzy soft coefficients expressing uncertainty with respect to parameters in E.

To solve such equations, the fuzzy soft coefficients are combined with the variables according to the operations defined for fuzzy soft sets. After applying the corresponding fuzzy soft arithmetic and parameterization, the fuzzy soft differential equation is transformed into a family of crisp separable differential equations:

$F_e\ \phi(y)dy = G_e\ \psi(\mathrm{x})$ dx, e∈E,

Where F_e and G_e denoted the membership-valued coefficient associated with parameter e. Each resulting crisp equation is solved separately by integration

$\int F_e\ \phi(y)dy = \int G_e\ \psi(\mathrm{x})$ dx, e∈E.

The collection of all parameterized solutions constitutes the solution of the first-order separable differential equation with fuzzy soft set coefficients.

Type 1. Consider the differential equation

(F,E) $y\ dy$ =(G,E) x dx

Where

E=$\{e_1, e_2\}$

Is the parameter set, and the fuzzy soft coefficients are defined by

(F,E)=$\{ e_1 \to 2, e_2 \to 3\}$,

(G,E)=$\{ e_1 \to 4, e_2 \to 6\}$.

The equation is a first-order separable differential equation with fuzzy soft set coefficients because

- the variables x and y are separable,
- the coefficients are represented through fuzzy soft parameterization.

Step 1: Convert into Crisp Equations. For each parameter $e \in E$, obtain a crisp separable differential equation.

For e_1:

2y dy= 4x dx

Integrating both sides

$$\int 2\mathrm{y}\,\mathrm{dy} = \int 4\mathrm{x}\,\mathrm{dx}$$

$$y^2 = 2x^2 + C_1$$

For e_2:

3y dy= 6x dx

Integrating both sides

$$\int 3\mathrm{y}\,\mathrm{dy} = \int 6\mathrm{x}\,\mathrm{dx}$$

$$\frac{3}{2} y^2 = 3x^2 + C_2$$

Or equivalently,

$$y^2 = 2x^2 + C_2'$$

Hence, the family of solutions corresponding to the fuzzy soft parameters is
$(S, E) = \{ e_1 \to y^2 = 2x^2 + C_1, e_2 \to y^2 = 2x^2 + C_2'\}$.
This collection represents the solution of the first-order separable differential equation with fuzzy soft set coefficients.

$$y(x) = \begin{cases} \{e_1 \to \sqrt{2x^2 + C_1}, e_2 \to \sqrt{2x^2 + C_2'}\}, x \geq 0, \\ \{e_1 \to -\sqrt{2x^2 + C_1}, e_2 \to -\sqrt{2x^2 + C_2'}\}, x \leq 0, \end{cases}$$

Type 2. Consider the fuzzy soft separable differential equation

$$y' = \frac{(G, E)y^2}{(F, E)x},$$

Where the parameter set is
$E = \{e_1, e_2, e_3\}$.
Assume the fuzzy soft coefficients are
$(F, E) = \{e_1 \to 2, e_2 \to 4, e_3 \to 6\}$,
$(G, E) = \{e_1 \to 3, e_2 \to 5, e_3 \to 7\}$.
Thus

$$y' = \frac{\langle 3,5,7\rangle y^2}{\langle 2,4,6\rangle x}.$$

Step 1: Division of Fuzzy Soft Coefficients. Divide the fuzzy coefficients

$$\frac{\langle 3,5,7\rangle}{\langle 2,4,6\rangle} = \langle \frac{3}{2}, \frac{5}{4}, \frac{7}{6}\rangle$$

Hence,

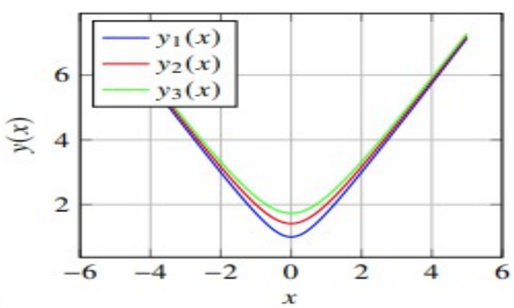

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FIGURE 1. Classical graphs

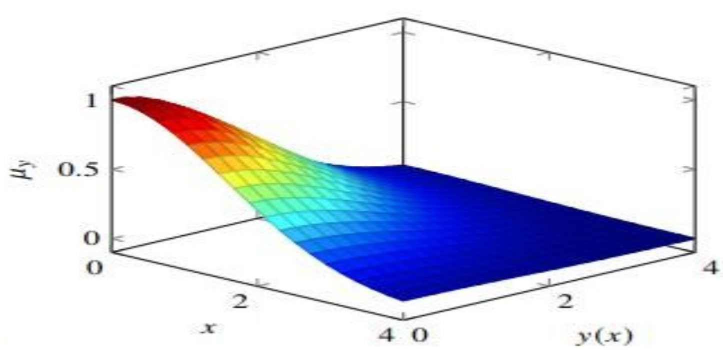

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FIGURE 2. Fuzzy form

$$\frac{dy}{dx} = \frac{\langle 3,5,7\rangle y^2}{\langle 2,4,6\rangle x}.$$

Step 2: Fuzzy Multiplication. Applying fuzzy multiplication,

$$\frac{dy}{dx} = \{\min\left(\frac{3}{4}\frac{y^2}{x}, \frac{7}{3}\frac{y^2}{x}\right), \frac{5}{4}\frac{y^2}{x}, max\left(\frac{3}{4}\frac{y^2}{x}, \frac{7}{3}\frac{y^2}{x}\right)\}.$$

Since $y^2 > 0$, the sign depends only on x.

For $x \geq 0$,

$$\frac{dy}{dx} = \langle \frac{3}{4}\frac{y^2}{x}, \frac{5}{4}\frac{y^2}{x}, \frac{7}{3}\frac{y^2}{x} \rangle.$$

For $x < 0$,

$$\frac{dy}{dx} = \langle \frac{7}{3}\frac{y^2}{x}, \frac{5}{4}\frac{y^2}{x}, \frac{3}{4}\frac{y^2}{x} \rangle.$$

Step 3: Convert into Crisp Separable Differential Equations. For $x \geq 0$, obtain

$$\frac{dy}{dx} = \frac{3}{4}\frac{y^2}{x},$$
$$\frac{dy}{dx} = \frac{5}{4}\frac{y^2}{x},$$
$$\frac{dy}{dx} = \frac{7}{3}\frac{y^2}{x},$$

Step 4: Solve Each Equation
First Equation.

$$\frac{dy}{dx} = \frac{3}{4}\frac{y^2}{x}$$

Separate Variables,

$$y^2 dy = \frac{3}{4} x\, dx$$

Integrate:

$$\int y^2 dy = \int \frac{3}{4} x\, dx,$$
$$\frac{y^3}{3} = \frac{3}{8} x^2 + C_1$$

Thus
$8y^3 = 9x^2 + C_1$.
Second Equation.

$$\frac{dy}{dx} = \frac{5}{4}\frac{y^2}{x}$$
$$y^2 dy = \frac{5}{4} x\, dx$$

Integrate:

$$\frac{y^3}{3} = \frac{5}{8} x^2 + C_2$$

Thus
$8y^3 = 15x^2 + C_2$.
Third Equation.

$$\frac{dy}{dx} = \frac{7}{3}\frac{y^2}{x}$$
$$y^2 dy = \frac{7}{3} x\, dx$$

Integrate:

$$\frac{y^3}{3} = \frac{7}{6} x^2 + C_3$$

Thus
$2y^3 = 7x^2 + C_3$.
Final fuzzy soft solution. For $x \geq 0$,

$$\{8y^3 = 9x^2 + C_1, 8y^3 = 15x^2 + C_2, 2y^3 = 7x^2 + C_3\}$$

For $x < 0$,
$\{2y^3 = 7x^2 + C_4, 8y^3 = 15x^2 + C_5, 8y^3 = 9x^2 + C_6\}$.
These represent the fuzzy soft solutions of the given first-order separable differential equation.

$$y(x) = \begin{cases} \{\sqrt[3]{\frac{9x^2 + C_1}{8}}, \sqrt[3]{\frac{15x^2 + C_2}{8}}, \sqrt[3]{\frac{7x^2 + C_3}{2}}\}, x \geq 0, \\ \{\sqrt[3]{\frac{7x^2 + C_4}{2}}, \sqrt[3]{\frac{15x^2 + C_5}{8}}, \sqrt[3]{\frac{9x^2 + C_6}{8}}\}, x \leq 0, \end{cases}$$

Type 3.
Given equation.

$$y' = \frac{(G,E)}{(F,E)}y + x,$$

Where
$E = \{e_1, e_2, e_3\}$, $(F,E) = \{1,2,3\}$, $(G,E) = \{2,4,6\}$
So fuzzy ratio becomes

$$\frac{\langle 2,4,6\rangle}{\langle 1,2,3\rangle} = \langle 2,2,2\rangle = 2$$

Hence the crisp form is

$$y' = 2y + x,$$

Step 1: convert to standard linear form.

$$y' - 2y = x$$

Step 2: Find the integrating factor.

$$IF = e^{\int -2dx} = e^{-2x}$$

Step 2: Multi throughout.

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}x$$

Left side becomes:

$$\frac{d}{dx}(ye^{-2x}) = xe^{-2x}$$

Diagram
Step 4: Integrate

$$ye^{-2x} = \int xe^{-2x}dx$$

Using integrate by parts
Let

$$u = x, dv = e^{-2x}dx$$

Then:

$$\int xe^{-2x}dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}$$

So

$$ye^{-2x} = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$

Step 5: Final solution. Multiply by e^{2x}

$$y = -\frac{x}{2} - \frac{1}{4} + Ce^{2x}.$$

Closure Under Addition Property. If (F,E) and (G,E) are fuzzy soft solutions of FSDEs, then their sum (F⊕G)(e,t) is also a fuzzy soft set.

Proof. Using α cut representation, $[F(e,t)]^\alpha = [\underline{F}^\alpha, \overline{F}_\alpha]$ and $[G(e,t)]^\alpha = [\underline{G}^\alpha, \overline{G}_\alpha]$. Addition of intervals gives $(F \oplus G)^\alpha = [\underline{F}^\alpha + \underline{G}^\alpha, \overline{F}_\alpha + \overline{G}_\alpha]$. Since $\underline{F}^\alpha + \underline{G}^\alpha \leq \overline{F}_\alpha + \overline{G}_\alpha$, the result again a valid fuzzy interval. Hence (F⊕G,E) is fuzzy soft set.

Closure Under Scalar Multiplication Property. For any scalar k∈ ℝ, the fuzzy soft scalar product k(F,E) is also a fuzzy soft set.

Proof. For a triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$, scalar multiplication gives $k\tilde{a} = (ka_1, ka_2, ka_3)$**.** Thus, $k[F(e,t)]^\alpha = [k\underline{F}^\alpha, k\overline{F}_\alpha]$. which is again an interval. Therefore scalar multiplication preserves fuzzy softness.

Linearity Property. Consider the FSDE $\frac{dF}{dt} = A(e,t)F + B(e,t)$**.** If F_1 and F_2 are solutions of the homogeneous equation $\frac{dF}{dt} = A(e,t)F$, then $c_1F_1 + c_2F_2$ is also a solution.

Proof. Differentiate
$\frac{d}{dt}(c_1F_1 + c_2F_2) = c_1\frac{dF_1}{dt} + c_2\frac{dF_2}{dt}$.
Since F_1 and F_2, satisfy the equation,
$= c_1A(e,t)F_1 + c_2A(e,t)F_2$.
Factor A(e,t):
$= A(e,t)(c_1F_1 + c_2F_2)$.
Hence the linear combination is also a solution.

Existence Property. Suppose $G(e, t, F)$ is continuous in t and Lipschitz continuous in F. Then the FSDE $\frac{dF}{dt} = G(e,t,F)$ with the initial condition $F(e,t_0) = F_0(e)$ has at least one fuzzy soft solution.

Proof. Using α-cut representation,

$$\frac{d}{dt}\underline{F}^\alpha = \underline{G}^\alpha,$$

$\frac{d}{dt}F_\alpha = \overline{G}_\alpha$.
Each equation is an ordinary differential equation.
By the Picard–Lindelo¨f theorem, continuity and Lipschitz conditions guarantee existence of solutions. Therefore each α-cut has a solution, implying existence of the fuzzy soft solution.
Uniqueness Property. If $G(e, t, F)$ satisfies a Lipschitz condition in F, then the FSDE solution is unique.
Proof. Suppose F_1 and F_2 are two solutions. Then
$\frac{d}{dt}(F_1 - F_2) = G(e,t,F_1) - G(e,t,F_2)$.
Using the Lipschitz condition,
$|G(e,t,F_1) - G(e,t,F_2)| \leq L|F_1 - F_2|$.
Applying Gro¨nwall's inequality gives
$F_1 = F_2$.
Hence the solution is unique.
Reduction Property. Every fuzzy soft differential equation reduces to a family of crisp differential equations.
Proof. Using α-cuts,
$[F(e,t)]^\alpha = [\underline{F}^\alpha(e,t), \overline{F}_\alpha(e,t)]$.
Thus

$$\frac{d}{dt}[F(e,t)]^\alpha = [\frac{d}{dt}\underline{F}^\alpha, \frac{d}{dt}\overline{F}_\alpha].$$

Therefore the FSDE decomposes into ordinary interval differential equations. Hence solving the FSDE is equivalent to solving a parameterized family of crisp ODEs.

Parameter Independence. If parameters $e_i, e_j \in E$ are independent, then their corresponding crisp differential equations can be solved separately.
Proof. For each parameter $e \in E, \frac{dF_e}{dt} = G_e(t, F_e)$ depends only on the corresponding parameter. Thus equations for different parameters do not interact. Hence each parameterized equation is independently solvable.
Stability Property. Consider $\frac{dF}{dt} = -\lambda F,\ \lambda > 0$. Then the fuzzy soft solution is asymptotically stable.

Proof. The solution is
$F(e,t) = F_0(e)e^{-\lambda F}$,
As $t \to \infty$,
$e^{-\lambda F} \to 0$,
Therefore
$F(e,t) \to 0$.
Hence the equilibrium solution is asymptotically stable.
Superposition Property. If $F_1, F_2, \ldots, F_n$ are solutions of a linear homogeneous FSDE, then $\sum_{i=1}^{n} c_iF_i$ is also a solution
Proof. By linearity,

$$\frac{d}{dt}(\sum_{i=1}^{n} c_iF_i) = \sum_{i=1}^{n} c_i \frac{d}{dt}F_i = \sum_{i=1}^{n} c_iAF_i = A\sum_{i=1}^{n} c_iF_i.$$

Hence the superposition is also a solution.
Consistency with Classical Differential Equations. If all fuzzy soft coefficients become crisp singleton values, then the FSDE reduces to a classical differential equation.

Proof. Suppose $(F, E) = \{e \to a\}$ where a is crisp. Then the α-cut becomes $[a, a]$.Thus $\underline{F}^\alpha = \overline{F}_\alpha$. Hence the interval system collapses into a single ordinary differential equation. Therefore the FSDE reduces to the classical ODE.

ConcLUsion

This study presents a systematic framework for solving first-order differential equations using fuzzy soft set theory. By incorporating parameter uncertainty through fuzzy soft coefficients, classical differential equations are extended into fuzzy soft differential models that better represent real-world uncertainty. The proposed methodology demonstrates that fuzzy soft differential equations can be effectively transformed into crisp differential equations by fixing parameter values from the parameter set. This reduces the problem into a family of deterministic differential equations, each corresponding to a specific parameter. The resulting solutions collectively form the fuzzy soft solution set, preserving uncertainty in a structured manner. -cut representation plays a crucial role in interpreting fuzzy solutions by converting fuzzy-valued functions into interval-valued systems. This enables the application of standard analytical techniques such as separation of variables, integrating factors, and linear differential equation methods. Three illustrative types of fuzzy soft differential equations were analyzed Type 1: Separable equations with fuzzy soft coefficients leading to parameter-dependent classical solutions. Type 2: Nonlinear fuzzy soft separable equations involving fuzzy ratio operations and piecewise solution structures. Type 3: Linear fuzzy soft differential equations reduced to crisp linear ODEs using fuzzy ratio simplification. Across all cases, the results confirm that fuzzy soft set theory provides a powerful and flexible tool for handling uncertainty in differential equations while maintaining compatibility with classical solution techniques.

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