Planarity And Energy of Bijective Fuzzy Subgraphs

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Abstract
In this paper a concept of Fuzzy Planar Subgraphs and the relation between Edge Deletion Bijective Fuzzy Subgraph of Considerable and Effective Edges and its Fuzzy Planarity value, size have been discussed. Some results on Energy and Laplacian Energy of Edge Deletion Bijective Fuzzy Subgraph have been found. Various cases focusing on the Energy and Laplacian Energy of the graph K_5 is explored.

Keywords: Bijective fuzzy planar graph, Edge deletion fuzzy subgraph, Fuzzy energy and laplacian energy.

1.Introduction

Graph theory is an important branch of discrete mathematics with applications in computer science, engineering, communication networks, and transportation systems. A graph consists of vertices and edges that represent relationships between objects. The basic concept of graph theory such as connectivity, planarity, and subgraphs are well explained in standard texts like Balakrishnan's Graph Theory [3]. However, many real-life systems involve uncertainly, where relationships may exist with different strengths. This limitation led to the development of fuzzy graph theory. The concept of fuzzy graphs was introduced by Rosenfeld in1975 [16]. In a fuzzy graph, vertices and edges are assigned membership values in the interval [0,1], representing the degree of connection between elements. Abdul-Jabbar et al [1] studied fuzzy dual graphs, McAllister [10] worked on fuzzy intersecting graphs, and Samanta and Pal [17] introduced fuzzy threshold graphs. These works strengthened the structure of fuzzy graph theory. Fuzzy planar graphs form an important extension of fuzzy graphs. In classical graph theory, a graph is planar if it can be drawn without edges crossings. Extending this idea, Samanta and Pal [18] developed the theory of fuzzy planar graph fuzzy graph configuration were discussed by Nirmala and Dhanabal [11], and interval-valued fuzzy planar graphs were introduced by Pramanik, Samanta, and Pal [13]. Applications of fuzzy planar graphs include VLSI network partitioning models by Kalathian et al [7] and smart city traffic models by Rajalaxmi [15]. These studies show that fuzzy planar graphs are useful in practical network design. Another important area is spectral graph theory, graph energy is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix. Energy of different graphs including regular [5], non-regular [6], circular [20] and random graphs [4] is also under study. The concept was extended to fuzzy graphs by Anjali and Mathew [2] and Laplacian energy of fuzzy graphs was introduced by Rahimi and Fayazi [14]. Lauri [8] studied vertex-deleted and edge-deleted subgraphs, and Lu, Wang, and Bai [9] further analyzed structural properties related to deletion. When applied to fuzzy planar graphs. In this paper, fuzzy planar subgraphs, its planarity, energy and Laplacian energies were explored.

2. PRELIMINARIES

Definition 2.1 [15]
Let $G' = (V, \sigma, \mu')$ be the Considerable Bijective Fuzzy Graph and $E' = \mu'(x,y){:}\, \mu'(x,y) \neq 0, \forall x,y \in V$. Let U' be the subset of E' such that the deletion of the set U' from E' such that the deletion of the set U′ from E′forms a subgraph called ED-fuzzy subgraph of $G.G' = \mu'(x,y) \in E : \mu'(x,y) \neq 0, \forall x,y \in V$.The fuzzy graph$G' - U'$ is a Considerable edge deletion bijective fuzzy subgraph $(CEDBFSG)$ of$G$.

Definition 2.2[15]
Let $G'' = (V, \sigma, \mu')$ be the Effective Bijective Fuzzy Graph and $E'' = \mu''(x,y){:}\, \mu''(x,y) \neq 0, \forall x,y \in V$. Let U'' be the subset of E'' such that the deletion of the set U'' from E'' such that the deletion of the set U″from E″forms a subgraph called ED-fuzzy subgraph of G. $G'' = \mu''(x,y) \in E : \mu''(x,y) \neq 0, \forall x,y \in V$.The fuzzy graph$G'' - U''$ is an Effective edge deletion bijective fuzzy subgraph $(EEDBFSG)$ of$G$.

Definition 2.3 [2]
The *adjacency matrix*A of a fuzzy graph$G = (V, \sigma, \mu)$is an n matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu(v_i, v_j)$. Note that A becomes that usual adjacency matrix when all the non-zero membership values are 1.

Definition 2.4 [2]
Let $G = (V, \sigma, \mu)$ be a fuzzy graph and A be its adjacency matrix. The eigenvalues of A are called *eigenvalues* of G. The spectrum of A is called the spectrum of G.

Definition 2.5[2]

Let $G = (V, E)$ be fuzzy graph. Adjacency matrix be $A(G)$ and eigenvalue of A is λ_i. Fuzzy Energy denoted by $E(G)$.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

Definition 2.6 [14]
Let u be a vertex of the fuzzy graph $G = (\sigma, \mu)$. The degree of u is defined as $d_G(u) = \sum \mu(uv)$.
Definition 2.7[14]
Let $A(G)$ be an adjacency matrix and $D(G) = [d_{ij}]$ be a degree matrix of $G = (\sigma, \mu)$. The matrix $L(G) = D(G) - A(G)$ is defined as fuzzy Laplacian matrix of G.

THEOREM 2.8
If$G$is a Bijective fuzzy graph and G' is a Considerable edge bijective fuzzy subgraph of G then $f' \leq f$.
PROOF:
Let $G = (V, \sigma, \mu)$ be a Bijective fuzzy graph (BFG) and $G' = (V, \sigma, \mu')$ be considerable edge deletion bijective fuzzy subgraph $(CEDBFSG)$ of G,Where all the vertex and edge are both injective and surjective. V is the vertex set, $\sigma: V \to [0,1]$ is the vertex membership function and$\mu$is edge set, $\mu: V \times V \to [0,1]$ is the edge membership function. Here vertex set remains constant and its membership function will become as $\sigma_{G'}(V) = \sigma_G(V)$. The fuzzy planarity value of G is f, the fuzzy planarity value with effective number $c \geq 0.5$ for any effective edge$I_{(a,b)} \geq 0.5$ after removing considerable edge. The fuzzy planarity value of G' is f',there is no new edge added and crossing cannot be done. The fuzzy planarity value of G will be less than or equal to fuzzy planarity value of G'. Hence, $f' \leq f$.
THEOREM 2.9
If G is a bijective fuzzy graph and G'' is Effective edge bijective fuzzy subgraph of G then$f'' \geq f$.
PROOF:
Let $G = (V, \sigma, \mu)$ be a Bijective fuzzy graph (BFG) andG$'' = (V, \sigma, \mu'')$ be effective edge deletion bijective fuzzy subgraph $(EEBFSG)$ of G, where all the vertex and edge are both injective and surjective. V is the vertex set, $\sigma: V \to [0,1]$ is the vertex membership function and$\mu$is edge set, $\mu: V \times V \to [0,1]$ is the edge membership function. Here vertex set remains constant and its membership function will become as $\sigma_{G''}(V) = \sigma_G(V)$. The fuzzy planarity value of G is f, the fuzzy planarity value with considerable number $c < 0.5$ for any considerable edge$I_{(a,b)} < 0.5$ after removing effective edge. The fuzzy planarity value of $G''$is f'',there is no new edge added and crossing cannot be done. The fuzzy planarity value of G will be less than or equal to fuzzy planarity value of G''. Hence, $f'' \geq f$.
THEOREM 2.10
If G is a Bijective fuzzy graph and G' is a Considerable edge bijective fuzzy subgraph and G'' is a Effective edge bijective fuzzy subgraph of G then $f'' \geq f'$.
PROOF:
If G is a BFG and G' is a Considerable edge bijective fuzzy subgraph of G then $f' \leq f$ $-----(1)$
If G is a BFG and G'' is an Effective edge bijective fuzzy subgraph of G then $f'' \geq f$ $-----(2)$
From (1) and (2) the statement is obtained.
THEOREM 2.11
If G is Bijective fuzzy graph and G' be considerable edge deletion bijective fuzzy subgraph, then $E(G) \geq E(G')$.
PROOF:
Let $G = (V, E)$ be a Bijective fuzzy graph (BFG) and $G' = (V, E')$ of considerable edge deletion bijective fuzzy subgraph $(CEDBFSG)$ of G. Adjacency matrix $A(G)$ of G we get eigenvalues $\lambda_i's$ then, $E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. Adjacency matrix $A(G')$ of G' we get eigenvalues $\lambda_i's$ then, $E(G') = \sum_{i=1}^{n} |\lambda_i(G)|$. $A(G') = A(G) - B$ where B is matrix since only considerable edges are present.$\sum_{i=1}^{n} |\lambda_i(G')| \leq \sum_{i=1}^{n} |\lambda_i(G)|$where, $E(G') = \sum_{i=1}^{n} |\lambda_i(G')|$ and $E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. Hence, $E(G) \geq E(G')$.
THEOREM 2.12
Let G be Bijective fuzzy graph and G'' be the eective subgraph of G, then $E(G) \geq E(G'')$.
PROOF:
Let $G = (V, E)$ be a Bijective fuzzy graph (BFG) and $G'' = (V, E'')$ of effective edge deletion bijective fuzzy subgraph $(EEDBFSG)$ of G. Adjacency matrix $A(G)$ of G we get eigenvalues $\lambda_i's$ then, $E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. Adjacency matrix $A(G'')$ of G'' we get eigenvalues $\lambda_i's$ then, $E(G') = \sum_{i=1}^{n} |\lambda_i(G)|$. $A(G'') = A(G) - B$ where B is matrix since only considerable edges are present.$\sum_{i=1}^{n} |\lambda_i(G'')| \leq \sum_{i=1}^{n} |\lambda_i(G)|$ where, $E(G'') = \sum_{i=1}^{n} |\lambda_i(G')|$ and $E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. Hence, $E(G) \geq E(G'')$.

THEOREM 2.13

If G is a Bijective fuzzy graph, G' is it's be considerable subgraph of G and G'' is its effective subgraph of G then Energy with effective edges is always more than Energy without effective edges.

PROOF:

Let $G = (V, E)$ be a Bijective fuzzy graph (BFG) and $G' = (V, E')$ of considerable edge deletion bijective fuzzy subgraph $(CEDBFSG)$ of G. $G'' = (V, E'')$ of effective edge deletion bijective fuzzy subgraph $(EEDBFSG)$ of G Adjacency matrix $A(G')$ of G' we get eigenvalues $\lambda_i's$ then, $E(G') = \sum_{i=1}^{n} |\lambda_i(G)|$. Adjacency matrix $A(G')$ of G' we get eigenvalues $\lambda_i's$ then, $E(G') = \sum_{i=1}^{n} |\lambda_i(G)|$ Adjacency matrix $A(G'')$ of G'' we get eigenvalues $\lambda_i's$ then, $E(G'') = \sum_{i=1}^{n} |\lambda_i(G)|$ then, $\sum_{i=1}^{n} |\lambda_i(G')| \leq \sum_{i=1}^{n} |\lambda_i(G'')|$ where, $E(G'') \sum_{i=1}^{n} |\lambda_i(G'')|$ and $E(G') \sum_{i=1}^{n} |\lambda_i(G')|$. Hence, $E(G'') \geq E(G')$.

THEOREM 2.14

If Energy of Bijective fuzzy graph is $E(G)$ and Laplacian energy of bijective fuzzy graph is $LE(G)$, then $E(G) \leq LE(G)$.

PROOF:

Let $G = (V, E)$ be a bijective fuzzy graph (BFG) and $LE(G)$ be Laplacian energy of G. Adjacency matrix $A(G)$ of G we get eigenvalues $\lambda_i's$ then, $E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. Adjacency matrix $A(G)$ of G we get eigenvalues $\mu_i's$ then, $LE(G) = \sum_{i=1}^{n} |\mu_i(G)|$.$LE(G) = D(G) - A(G)$where $D(G)$ is degree matrix of G. $\sum_{i=1}^{n} |\mu_i(G)| \geq \sum_{i=1}^{n} |\lambda_i(G)|$ where, $LE(G) = \sum_{i=1}^{n} |\mu_i(G)|$ and$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$. Hence, $E(G) \leq LE(G)$.

THEOREM 2.15

Let G be Bijective fuzzy graph and G' be Effective subgraph of G then, $LE(G) \geq LE(G')$.

PROOF:

Let $G = (V, E)$ be a Bijective fuzzy graph (BFG) and $G' = (V, E')$ of considerable edge deletion bijective fuzzy subgraph $(CEDBFSG)$ of G. Adjacency matrix $A(G)$ of G we get eigenvalues $\mu_i's$ then, $LE(G) = \sum_{i=1}^{n} |\mu_i(G)|$, $LE(G) = D(G) - A(G)$ where $D(G)$ is degree matrix of G.Adjacency matrix $A(G')$ of G we get eigenvalues $\mu_i's$ then, $LE(G') = \sum_{i=1}^{n} |\mu_i(G')|$, $LE(G') = D(G') - A(G')$ where $D(G')$ is degree matrix of G.$\sum_{i=1}^{n} |\mu_i(G')| \leq \sum_{i=1}^{n} |\mu_i(G)|$where, $LE(G) = \sum_{i=1}^{n} |\mu_i(G)|$ and $E(G) = \sum_{i=1}^{n} |\mu_i(G)|$. Hence, $LE(G) \geq LE(G')$.

THEOREM 2.16

If G is a bijective fuzzy graph and G'' is its subgraph of G then, $LE(G) \geq LE(G'')$.

PROOF:

Let $G = (V, E)$ be a Bijective fuzzy graph (BFG) and$G' = (V, E')$ and $G'' = (V, E'')$of effective edge deletion bijective fuzzy subgraph $(EEDBFSG)$ of G. Adjacency matrix $A(G)$ of G we get eigenvalues $\mu_i's$ then, $LE(G) = \sum_{i=1}^{n} |\mu_i(G)|$, $LE(G) = D(G) - A(G)$ where $D(G)$ is degree matrix of G.Adjacency matrix $A(G)$ of G we get eigenvalues $\mu_i's$ then, $LE(G'') = \sum_{i=1}^{n} |\mu_i(G'')|$, $LE(G'') = D(G'') - A(G'')$ where $D(G'')$ is degree matrix of G.$\sum_{i=1}^{n} |\mu_i(G'')| \geq \sum_{i=1}^{n} |\mu_i(G)|$where, $LE(G) = \sum_{i=1}^{n} |\mu_i(G)|$ and $LE(G'') = \sum_{i=1}^{n} |\mu_i(G'')|$. Hence, $LE(G) \leq LE(G'')$.

THEOREM 2.17

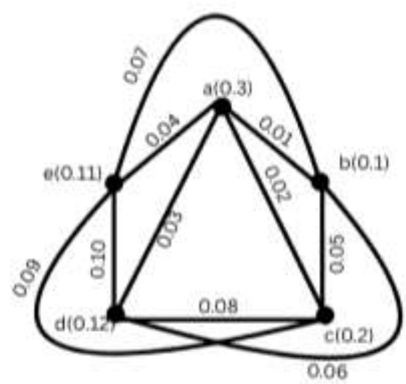
If G be Bijective fuzzy graph and G'be considerable subgraph of G with eigenvalues $\mu_i(G')$ and G'' be effective subgraph of G with eigenvalues $\mu_i(G'')$ then Laplacian Energy with effective edge is always more than Laplacian Energy without effective edge.

PROOF:

Let $G = (V, E)$ be a Bijective fuzzy graph (BFG) and $G' = (V, E')$ of considerable edge deletion bijective fuzzy subgraph $(CEDBFSG)$ of G. $G'' = (V, E'')$of effective edge deletion bijective fuzzy subgraph $(EEDBFSG)$ of G. Adjacency matrix $A(G')$ of G we get eigenvalues $\mu_i's$ then, $LE(G') = \sum_{i=1}^{n} |\mu_i(G')|$, $LE(G') = D(G') - A(G')$ where $D(G')$ is degree matrix of G and Adjacency matrix $A(G'')$ of G'' we get eigenvalues $\mu_i's$ then, $LE(G'') = \sum_{i=1}^{n} |\mu_i(G'')|$, $LE(G'') = D(G'') - A(G'')$ where $D(G'')$ is degree matrix of G.$\sum_{i=1}^{n} |\mu_i(G')| \leq \sum_{i=1}^{n} |\mu_i(G'')|$where, $LE(G'') = \sum_{i=1}^{n} |\mu_i(G'')|$ and $.LE(G') = \sum_{i=1}^{n} |\mu_i(G')|$. Hence, $LE(G'') \geq LE(G')$.

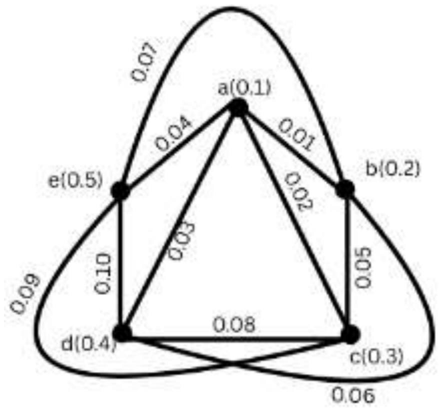
3. ENERGY AND LAPLACIAN ENERGY OF K_5

EXAMPLE 3.1



G_1: Bijective Fuzzy Graph

FIGURE 3.1
In G_1(b, e) and (b, d) are effective edges ,
$E(G_1)$ = 0.48
$E(G_1')$ = 0.35
$LE(G_1)$ = 1.1
$LE(G_1')$ = 0.88



G_1: Bijective Fuzzy Graph
FIGURE 3.2
In G_2 (b, e) and (b, d) are considerable edge ,
$E(G_2)$ = 0.48
$E(G_2'')$ = 0.35
$LE(G_2)$ = 1.1
$LE(G_2'')$ = 0.88
Comparing G_1 and G_2we get,

i. $E(G_1) = E(G_2) = 0.48$
ii. $E(G_1') = E(G_2'') = 0.35$
iii. $LE(G_1) = LE(G_2) = 1.1$
iv. $LE(G_1') = LE(G_2'') = 0.88$

APPLICATIONS IN LEVEL CROSSING

In level crossing two completely different paths were moving toward the single intersection point:

- The Police Van (Path A> 0.5):
  Anpolice transport van was racing down the highway toward the penitentiary. Van was operating perfectly under high security with 0.9(effective route) it had to cross the tracks to secure its transport.
- The Train (Path B< 0.5):
  The train was completely out of control at 80 miles per hour. Its safety value instantly crashed to 0.1(considerable route)

A disaster at the single crossing point seemed unavoidable. The police van could not take any alternate route, and the broken train could not stop. The intersection solved the difficulty because engineers built this compulsory intersection with robust network protocols, the crossing itself became an active safety shield. The millisecond the train's brake system dropped to 0.1, trackside sensors picked up the emergency error code.
The intersection's computer brain automatically took control of the highway. Even though the police van had a high priority (0.9), the system recognized that the train (0.1) could not stop. The police van driver saw the gates drop, applied the heavy brakes. Then, the train goes through the level crossing where it crossed the intersection without accident. Then police van stepped on the crossing the tracks reach the penitentiary gates right on time without any crash because the planarity value $f \geq 0.5$
Robust network design is not about avoiding intersections, one effective route and other considerable route crossing is allowed.there crossing of effective edge and considerable edge is safe than both are considerable or effective edges ($f < 0.5$)

Conclusion

In this dissertation, fuzzy planar subgraphs are studied. Edge Deletion Bijective Fuzzy Subgraphs between Considerable and Effective edges with its fuzzy planarity value is calculated. Also, Energy and Laplacian Energy of Edge Deletion Bijective Fuzzy Subgraph have been found. Various cases focusing on the Energy and Laplacian Energy of the K_5 is explored. In future Bijective fuzzy non planar graph of energy can be explored.

References

[1] Abdul-Jabbar, N., Naoom, J.H.,&Ouda, E. H. (2009). Fuzzy dual graph. *Journal of A1 Nahrain University*, 12(4), 168 − 171.
[2] Anjali, N., & Mathew, S. (2013). Energy of a fuzzy graph.

Annals of Fuzzy Mathematics and Informatics, 6(3), 455 − 465.
[3] Balakrishnan, V. K.(1977). *Graph theory*. McGraw-Hill.
[4] Du, W., Li, X., & Li, Y. (2010). Various energies of random graphs.*MATCH Communications in Mathematical and in Computer Chemistry*, 64(1), 251 − 260.
[5] Gutman, I., ZareFiroozabadi, S., de la Pena, J. A., & Rada, J.(2007). On the energy of regular graphs. *MATCH Communications in Mathematical and in Computer Chemistry, 57,*435 − 442.
[6] Indulal, G., &Vijayakumar, A. (2007). Energies of some non-regular graphs. Journal of Mathematical Chemistry, 42, 377 − 386.
[7] Kalathian, S., Ramalingam, S., &Deivanayagampillai, N. (2023). A fuzzy planar subgraph formation model for partitioning very large- scale integration networks. *Decision Analytics Journal*, 9,100339.
[8] Lauri, J. (1992). Vertex- deleted and edge-deleted subgraphs. In R. Ellul-Micallef& S. Fiorini (Eds.). *A collection of papers by members of the University of Malta on the occasion of its quarter centenary celebrstions.* Malta.
[9] Lu, H., Wang, W., & Bai, B. (2011). Vertex-deleted subgraphs and regular factors from regular graph. *Discrete Matheamtics*, 311(18 − 19), 2044 − 2048.
[10] McAllister, M. L. N. (1988). Fuzzy intersection graphs. *Computers & Mathematics with Applications*, 15(10),871 − 886.
[11] Nirmala, G., &Dhanabal, K. (2012). Special planar fuzzy graph configurations. *International Journal of Scientific and Research Publications*, 2(7),1 − 4.
[12] Pal, A., Samanta, S., & Pal, M. (2013). Concept of fuzzy planar graphs. In *Proceedings of the* 2013 *Science and Information Conference* (pp.557 − 563). IEEE.
[13] Pramanik, T., Samanta, S., & Pal, M. (2016). Interval-valued fuzzy planar graphs. *International Journal of Machine Learning and Cybernetics*, 7(4), 653 − 664.
[14] Rahimi, S. S., &Fayazi, F. (2014). Laplacian energy of a fuzzy graph.
[15] Rajalaxmi, D. (2025). Bijective fuzzy planar and non-planar graph in smart city traffic network. *International Journal of Applied Mathematics*, 38(7*s*), 265 − 272.
[16] Rosenfeld, A. (1975). Fuzzy graphs. In *uzzy sets and their applications to cognitive and decision processes* (pp. 77-95). Academic Press.
[17] Samanta, S., & Pal, M. (2011). Fuzzy threshold graphs. *CIIT International Journal of Fuzzy Systems*, 3(12), 360 − 364.
[18] Samanta, S., & Pal, M. (2015). Fuzzy planar graphs.*IEEE Transactions on Fuzzy Systems* , 23(6), 1939 − 1942.
[19] Samanta, S., Pal, M., & Pal, A. (2014). New concepts of fuzzy planar graph. *international Journal of Advanced Research in Artifical Intelligence*, 3(1), 52 − 59.
[20] Shparlinski, I. (2006). On the energy of some circular graphs. *linaer Algebra and Its Applications*, 414,378 − 382.