Common Fixed Point Theorems for Multi-Fuzzy Mappings via Weak Contractive Conditions in MR-Metric Spaces

A.N. Lakshmi Sudha[1], **Dr. T. Rakesh Singh**[2]

[1]Department of Mathematics, Rishi UBR Women's College, Hyderabad, Telangana, India rvsudha09@gmail.com
[2]Department of Mathematics, Aurora's Scientific and Technological Institute, Hyderabad, Telangana, India. rakeshsinghthakur16@gmail.com

Abstract
In this paper, we study common fixed point results for multiple fuzzy mappings in the framework of MR-metric spaces. By applying a max-type contractive con-dition, we establish the existence and uniqueness of common fixed points for four and six fuzzy mappings in a complete MR-metric space. Using a cyclic iterative construction together with the Hausdorff MR-metric structure, we prove the con-vergence of the generated sequence and demonstrate that all the mappings admit a unique common fixed point. The proposed results extend and strengthen existing fixed point principles by relaxing contraction assumptions and expanding the the-ory to multi-mapping settings. These findings further advance the development of fixed point theory in generalized metric spaces and provide a broader foundation for applications in fuzzy analysis, optimization, and uncertainty modeling

Keywords: Fixed point, fuzzy mapping, MR-metric space, MR-metric, contraction principle.

1 Introduction And Preliminaries

Fixed point theory is a branch of nonlinear analysis that studies conditions under which functions (or mappings) f have fixed points, defined as points x such that $f(x) = x$. It is a powerful tool for proving the existence and uniqueness of solutions to differential equations, integral equations, and in optimization problems.

The Banach contraction principle([4], [5]), which guarantees a unique fixed point for con-traction mappings([3]) on complete metric spaces, and the Brouwer fixed-point theorem([6]), which guarantees at least one fixed point for continuous functions mapping a compact convex set to itself. It serves as a, cornerstone in functional analysis, connecting topo-logical, geometric, and analytical methods. The theory provides iterative methods to approximate fixed points in various spaces, such as Banach or Hilbert spaces.The theory is widely applied in economics (Nash equilibrium), physics, engineering, and numerical analysis to determine stability and equilibria.

In the Theorem 2.1[1] authors the proved the existences and uniqueness of the fixed point for a mappings in MR-metric spaces, in theorem 2.3 [1] authors proved the existences and uniqueness of the fixed point for two mapping in MR-metric spaces by establishing the contraction property. In this paper we extend the results for Four and six mappings

Definition 1.1 (Fuzzy Set [7]). *Let X be a nonempty set. A fuzzy set A in X is defined by a membership function*
$\mu_A : X \to [0, 1]$,
where $\mu_A(x)$ *represents the degree of membership of x in A.*

Definition 1.2 (Fuzzy Mapping [10]). *Let* X *be a nonempty set. A mapping* $T : X \to F(X)$ *is called a fuzzy mapping if for each* $x \in X$, $T(x)$ *is a fuzzy set in* X.

Definition 1.3. *A point* $x \in X$ *is a common fixed point of* $T_1, T_2, \ldots, T_n$ *if*
$$x \in \bigcap_{i=1}^{n} [T_i(x)]_\alpha.$$

Definition 1.4 (Hausdorff MR-metric [9]). *Let* (X, M) *be an MR-metric space and* $A, B \subset X$ *be nonempty bounded sets.*
The Hausdorff MR-metric between A and B is defined by
$$H_M(A, B) = \max \left\{ \sup_{a\in A} \inf_{b\in B} M(a, b, =),\ \sup_{b\in B} \inf_{a\in A} M(a, b, =) \right\}.$$

Definition 1.5. *([2],[8]) Consider a non-empty set* $X \neq \emptyset$ *and a real number* $R > 1$. *A function* $M : X \times X \times X \to [0, \infty)$ *is termed an* MR-metric *if it satisfies the following conditions for all* $\upsilon, \xi, \zeta \in X$:
(M1) $M(\upsilon, \xi, \zeta) \geq 0$.
(M2) $M(\upsilon, \xi, \zeta) = 0$ *if and only if* $\upsilon = \xi = \zeta$.

(M3) $M(\upsilon, \xi, \zeta)$ *remains invariant under any permutation* $p(\upsilon, \xi, \zeta)$, *that is,* $M(\upsilon, \xi, \zeta) = M\big(p(\upsilon, \xi, \zeta)\big)$.

(M4) The following inequality holds: $M(\upsilon, \xi, \zeta) \leq R\left[M(\upsilon, \xi, \ell_1)+M(\upsilon, \ell_1, \zeta)+M(\ell_1, \xi, \zeta)\right]$.

A structure (X, M) *that adheres to these properties is called an* MR-metric space.

Definition 1.6. *An MR-metric space* (X, M) *is said to be complete if every Cauchy sequence in* (X, M) *converges to a point in* X.

2 Main Results

Theorem 2.1. *Let* (X, M) *be a complete MR-metric space with* $R > 1$. *Let* $T_1, T_2, T_3, T_4 : X \to F(X)$ *be four fuzzy mappings, where* $F(X)$ *denotes the family of all fuzzy subsets of* X.
Suppose there exists a constant $k \in (0, 1)$ *such that for all* $x, y \in X$ *and for each* $i = 1, 2, 3, 4$ *(with* $T_5 = T_1$*), the following cyclic contractive condition holds:*

$$H_M\big(T_i(x), T_{i+1}(y)\big) \leq k \max\big\{M(x, y, =), M\big(x, T_i(x), =\big), M\big(y, T_{i+1}(y), =\big)\big\}.$$

Then T_1, T_2, T_3, *and* T_4 *have a unique common fixed point in* X.

Proof. Let $x_0 \in X$ be arbitrary. Since $T_1(x_0)$ is a fuzzy subset of X, there exists $\alpha_0 \in (0, 1]$ such that the α_0-cut $[T_1(x_0)]_{\alpha_0}$ is nonempty.

Choose $x_1 \in [T_1(x_0)]_{\alpha_0}$.

Similarly, since $T_2(x_1)$ is nonempty, choose $x_2 \in [T_2(x_1)]_{\alpha_1}$. Proceeding in this way, we define inductively

$$x_{4n+1} \in [T_1(x_{4n})]_{\alpha_{4n}},\ x_{4n+2} \in [T_2(x_{4n+1})]_{\alpha_{4n+1}},$$
$$x_{4n+3} \in [T_3(x_{4n+2})]_{\alpha_{4n+2}},\ x_{4n+4} \in [T_4(x_{4n+3})]_{\alpha_{4n+3}}, \text{ for all } n \in \mathbb{N}.$$

Thus, we construct a sequence $\{x_n\}$ in X such that
$x_{n+1} \in T_{i_n}(x_n)$,
where $i_n \equiv n + 1 \pmod 4$.
Using the contractive hypothesis, we have
$$H_M\big(T_i(x_n), T_{i+1}(x_{n+1})\big) \leq k \max\{M(x_n, x_{n+1}, =), M(x_n, T_i(x_n), =), M(x_{n+1}, T_{i+1}(x_{n+1}), =)\}.$$
Since $x_{n+1} \in T_i(x_n)$ and $x_{n+2} \in T_{i+1}(x_{n+1})$, it follows that
$$M(x_{n+1}, x_{n+2}, =) \leq H_M\big(T_i(x_n), T_{i+1}(x_{n+1})\big).$$
Therefore,
$$M(x_{n+1}, x_{n+2}, =) \leq k \max\{M(x_n, x_{n+1}, =), M(x_n, x_{n+1}, =), M(x_{n+1}, x_{n+2}, =)\}.$$
Hence,
$$M(x_{n+1}, x_{n+2}, =) \leq k\, M(x_n, x_{n+1}, =).$$
Now we prove by induction that
$$M(x_n, x_{n+1}, =) \leq k^n M(x_0, x_1, =) \quad \text{for all } n \in \mathbb{N}.$$
For $n = 0$, we have
$$M(x_0, x_1, =) = k^0 M(x_0, x_1, =),$$
since $k^0 = 1$. Thus, the inequality holds for $n = 0$. Assume that for some $n \geq 0$, the inequality holds, that is,
$$M(x_n, x_{n+1}, =) \leq k^n M(x_0, x_1, =).$$
From equation(1), we have
$$M(x_{n+1}, x_{n+2}, =) \leq k\, M(x_n, x_{n+1}, =).$$
Using the inductive hypothesis, we substitute
$M(x_n, x_{n+1}, =) \leq k^n M(x_0, x_1, =)$ into the above inequality. Hence,
$$M(x_{n+1}, x_{n+2}, =) \leq k \cdot k^n M(x_0, x_1, =).$$
Since $k \cdot k^n = k^{n+1}$, we obtain
$$M(x_{n+1}, x_{n+2}, =) \leq k^{n+1} M(x_0, x_1, =).$$
This proves that the inequality holds for $n + 1$.
$$M(x_n, x_{n+1}, =) \leq k^n M(x_0, x_1, =) \quad \text{for all } n \in \mathbb{N}. \tag{1}$$
In particular, since $0 < k < 1$, we have $k^n \to 0$ as $n \to \infty$, and hence
$M(x_n, x_{n+1}, =) \to 0$.
Now we prove $\{x_n\}$ is Cauchy sequence.
Let $m > n$. Using property (M4) of MR-metric, we obtain

$$M(x_n, x_m, =) \leq R \sum_{i=n}^{m-1} M(x_i, x_{i+1}, =).$$

Using the contraction estimate,

$$M(x_n, x_m, =) \leq R \sum_{i=n}^{m-1} k^i M(x_0, x_1, =).$$

Since $0 < k < 1$, the geometric series converges and

$$M(x_n, x_m, =) \leq R \frac{k^n}{1-k} M(x_0, x_1, =).$$

Taking $n \to \infty$, we get

$$M(x_n, x_m, =) \to 0. \quad (2)$$

Hence $\{x_n\}$ is a Cauchy sequence.
We will prove the existence of a Common Fixed Point.
Since (X, M) is complete and $\{x_n\}$ is a Cauchy sequence, there exists $x^* \in X$ such that
$x_n \to x^*$ as $n \to \infty$.
We now show that
$x^* \in T_1(x^*)$, $x^* \in T_2(x^*)$, $x^* \in T_3(x^*)$, $x^* \in T_4(x^*)$.

Showing $x^* \in T_1(x^*)$. Recall that the sequence was constructed as
$x_{4n+1} \in T_1(x_{4n})$.
Since $x_{4n} \to x^*$ and $x_{4n+1} \to x^*$, we apply the contractive condition:

$$H_M\left(T_1(x_{4n}), T_2(x_{4n+1})\right) \leq k \max\{M(x_{4n}, x_{4n+1}, =), M(x_{4n}, T_1(x_{4n}), =), M(x_{4n+1}, T_2(x_{4n+1}), =)\}.$$

From equation (2), we know
$M(x_n, x_{n+1}, =) \to 0$.
Moreover, since $x_{4n+1} \in T_1(x_{4n})$, we have
$M(x_{4n}, T_1(x_{4n}), =) \leq M(x_{4n}, x_{4n+1}, =) \to 0$. Similarly, Hence,

$$M(x_{4n+1}, T_2(x_{4n+1}), =) \leq M(x_{4n+1}, x_{4n+2}, =) \to 0.$$

$$H_M\left(T_1(x_{4n}), T_2(x_{4n+1})\right) \to 0.$$

Passing to the limit as $n \to \infty$, we obtain

Now showing $x^* \in T_2(x^*)$. Since
$H_M\left(T_1(x^*), T_2(x^*)\right) = 0$.

$x_{4n+2} \in T_2(x_{4n+1})$,
and both subsequences converge to x^*, a similar argument yields
$H_M\left(T_2(x^*), T_3(x^*)\right) = 0$.
Now showing $x^* \in T_3(x^*)$. Since
$x_{4n+3} \in T_3(x_{4n+2})$,
and both subsequences converge to x^*, we similarly obtain
$H_M\left(T_3(x^*), T_4(x^*)\right) = 0$.
Now showing $x^* \in T_4(x^*)$.
Since
$x_{4n+4} \in T_4(x_{4n+3})$,
and both subsequences converge to x^*, we obtain
$H_M\left(T_4(x^*), T_1(x^*)\right) = 0$.
From the above relations, we have
$H_M(T_1(x^*), T_2(x^*)) = 0$, $H_M(T_2(x^*), T_3(x^*)) = 0$,
$H_M(T_3(x^*), T_4(x^*)) = 0$, $H_M(T_4(x^*), T_1(x^*)) = 0$.
Thus,
$T_1(x^*) = T_2(x^*) = T_3(x^*) = T_4(x^*)$
in the Hausdorff MR-metric sense
Since $x_{4n+1} \in T_1(x_{4n})$ and $x_{4n+1} \to x^*$, and the Hausdorff distance between $T_1(x_{4n})$ and $T_1(x^*)$ tends to zero, we conclude

Similarly, Therefore,$x^* \in T_1(x^*)$.

$x^* \in T_2(x^*)$, $x^* \in T_3(x^*)$, $x^* \in T_4(x^*)$.
$x^* \in T_1(x^*) \cap T_2(x^*) \cap T_3(x^*) \cap T_4(x^*)$.
Hence, x^* is a common fixed point of T_1, T_2, T_3, and T_4.

We will prove the uniqueness of the fixed point.
Suppose that x^* and y^* are two common fixed points of T_1, T_2, T_3, and T_4. That is,
$x^* \in T_1(x^*) \cap T_2(x^*) \cap T_3(x^*) \cap T_4(x^*)$,
and
$y^* \in T_1(y^*) \cap T_2(y^*) \cap T_3(y^*) \cap T_4(y^*)$.
now we shall prove that $x^* = y^*$.
Using the cyclic contractive condition for T_1 and T_2, we have,
$H_M \ T_1(x^*), T_2(y^*) \ \le k \max\{M(x^*, y^*, =), M(x^*, T_1(x^*), =), M(y^*, T_2(y^*), =)\}$.
Since x^* and y^* are common fixed points, we have
$x^* \in T_1(x^*)$, $\qquad y^* \in T_2(y^*)$.
Hence,
$M(x^*, T_1(x^*), =) = 0$, $\qquad M(y^*, T_2(y^*), =) = 0$.
Therefore, the inequality reduces to
$H_M \ T_1(x^*), T_2(y^*) \ \le k\, M(x^*, y^*, =)$.
Since $x^* \in T_1(x^*)$ and $y^* \in T_2(y^*)$, by the definition of the Hausdorff MR-metric, we obtain
$M(x^*, y^*, =) \le H_M \ T_1(x^*), T_2(y^*)$.
Combining with the previous inequality gives
$M(x^*, y^*, =) \le k\, M(x^*, y^*, =)$.
Rearranging the inequality, we get
$(1 - k)\, M(x^*, y^*, =) \le 0$.
Since $k \in (0, 1)$, we have $1 - k > 0$. Thus, the above inequality implies
$M(x^*, y^*, =) = 0$.
By property (M2) of MR-metric spaces,
$M(x^*, y^*, =) = 0 \implies x^* = y^*$.
Therefore, the common fixed point of T_1, T_2, T_3, and T_4 is unique. □

Theorem 2.2. *Let (X, M) be a complete MR-metric space with $R > 1$. Let $T_1, T_2, T_3, T_4, T_5, T_6$ be six fuzzy mappings. Suppose there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$,*
the following cyclic contractive condition holds:
$H_M \ T_i(x), T_{i+1}(y) \ \le k \max\{M(x, y, =), M(x, T_i(x), =), M(y, T_{i+1}(y), =)\}$
for $i = 1, 2, 3, 4, 5, 6$, where $T_7 = T_1$. Then the mappings T_1, T_2, T_3, T_4, T_5, and T_6 have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Choose $x_1 \in T_1(x_0)$, $x_2 \in T_2(x_1)$, $x_3 \in T_3(x_2)$, $x_4 \in T_4(x_3)$, $x_5 \in T_5(x_4)$, $x_6 \in T_6(x_5)$. Continue inductively
$x_{6n+1} \in T_1(x_{6n})$, $x_{6n+2} \in T_2(x_{6n+1})$, $x_{6n+3} \in T_3(x_{6n+2})$, $x_{6n+4} \in T_4(x_{6n+3})$, $x_{6n+5} \in T_5(x_{6n+4})$, $x_{6n+6} \in T_6(x_{6n+5})$.
Thus, we obtain a sequence $\{x_n\}$ in X.
Using the cyclic contractive condition, we obtain
$M(x_{n+1}, x_{n+2}, =) \le k\, M(x_n, x_{n+1}, =)$.
Now, by induction, we get
$M(x_n, x_{n+1}, =) \le k^n M(x_0, x_1, =)$.
Since $0 < k < 1$, we have $k^n \to 0$. by using property (M4),

Hence,

$$M(x_n, x_m, =) \le R \sum_{i=n}^{m-1} M(x_i, x_{i+1}, =).$$

$$M(x_n, x_m, =) \le R \sum_{i=n}^{m-1} k^i M(x_0, x_1, =).$$

Since the geometric series converges,

$$M(x_n, x_m, =) \le R \frac{k^n}{1-k} M(x_0, x_1, =).$$

Thus $\{x_n\}$ is Cauchy.
Now we will prove existence of Common Fixed Point.
Since (X, M) is complete, there exists $x^* \in X$ such that
$x_n \to x^*$.
Using the same argument as in Theorem 2.3([1]) and passing to the limit in the cyclic inequality, we obtain
$H_M(T_1(x^*), T_2(x^*)) = 0$, $H_M(T_2(x^*), T_3(x^*)) = 0$, $H_M(T_3(x^*), T_4(x^*)) = 0$,

$H_M(T_4(x^*), T_5(x^*)) = 0, H_M(T_5(x^*), T_6(x^*)) = 0, H_M(T_6(x^*), T_1(x^*)) = 0.$
Thus, x^* is the fixed point.
Now we shall prove uniqueness of fixed point.
Suppose x^* and y^* are two common fixed points. Using the contractive condition for T_1 and T_2, we obtain
$M(x^*, y^*, =) \leq k\, M(x^*, y^*, =).$
Since $k \in (0, 1)$, it follows that
$M(x^*, y^*, =) = 0.$
By property (M2), $x^* = y^*$. Hence, x^* is unique.

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